# ON DISTINGUISHING REPRESENTATIONS OF DIFFERENT CONDUCTOR

Abstract. For infinite-dimensional irreducible admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ with trivial central character, we construct explicit Hecke algebra elements whose traces distinguish a given conductor from lower conductors. This answers a question of Masao Tsuzuki.

Denote by $G$ the group $\mathrm{GL}_2(\mathbb{Q}_p)$ and by $K$ the maximal compact subgroup $\mathrm{GL}_2(\mathbb{Z}_p)$. For $n \geq 0$, let

$$K_0(p^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K : c \in p^n\mathbb{Z}_p \right\}.$$

Thus $K_0(p^0) = K$, and $K_0(p)$ is the usual Iwahori subgroup. For each fixed $n$, normalize Haar measure $dg$ on $G$ by $\mathrm{vol}(K_0(p^n)) = 1$. Let $H(G//K_0(p^n))$ be the Hecke algebra of locally constant, compactly supported, $K_0(p^n)$-bi-invariant functions on $G$, with convolution

$$f_1 * f_2(h) = \int_G f_1(g) f_2(g^{-1}h)\, dg = \int_G f_1(hg) f_2(g^{-1})\, dg.$$

Let $H(K//K_0(p^n))$ be its subalgebra of functions supported on $K$.

For $i \geq 0$, let $X_i$ denote the set of infinite-dimensional irreducible admissible representations $\pi$ of $G$ with trivial central character such that $V_\pi^{K_0(p^i)} \neq 0$, taken up to isomorphism. Write $c_\pi$ for the conductor exponent of $\pi$, and set

$$X_i^* = \{\pi \in X_i : c_\pi = i\}.$$

Thus $X_n = \bigsqcup_{i \leq n} X_i^*$. Recall the following theorem.

**Theorem 1** (Casselman [2]). *Let $(\pi, V_\pi)$ be an infinite-dimensional irreducible admissible representation of $G$ with trivial central character and conductor exponent $c_\pi$. Then*

$$\dim V_\pi^{K_0(p^m)} = \max\{m - c_\pi + 1, 0\} \qquad (m \geq 0).$$

For $\pi \in X_n$, the algebra $H(K//K_0(p^n))$ acts on $V_\pi^{K_0(p^n)}$ by

$$\pi(\phi)v = \int_K \phi(g)\pi(g)v\, dg.$$

In this note we consider a question of Masao Tsuzuki, which may be phrased as follows: *Can explicit elements of $H(K//K_0(p^n))$ distinguish the fixed spaces $V_\pi^{K_0(p^n)}$ for representations of different conductor?* We use traces to distinguish representations of conductor exponent $n$ from those of lower conductor. More precisely, following our work in [1], we construct an explicit $\phi_n \in H(K//K_0(p^n))$ such that

$$\operatorname{tr}\pi(\phi_n) = 0 \quad (\pi \in X_i^*,\ i < n), \qquad \operatorname{tr}\pi(\phi_n) = 1 \quad (\pi \in X_n^*).$$

Throughout, this trace is taken on $V_\pi^{K_0(p^n)}$.

We recall some notation and results from [1]. In the following algebra calculations assume $n \geq 1$, and write $K_0 = K_0(p^n)$. Let $X_g$ denote the characteristic function of $K_0 g K_0$ for $g \in G$, and let $I = 1_{K_0}$, the identity of the Hecke algebra. Put $y(t) = \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$ and $w(1) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Let $\mathcal{U}_0 = X_{w(1)}$ and $\mathcal{V}_r = X_{y(p^r)}$ for $1 \leq r \leq n-1$. For $1 \leq r \leq n$, set $\mathcal{Y}_r = I + \sum_{j=r}^{n-1} \mathcal{V}_j$, so that $\mathcal{Y}_n = I$. The following relations are given in [1, Corollary 3.11, Proposition 3.12 and Theorem 6].

**Theorem 2.** *(1)* $\mathcal{Y}_r^2 = p^{n-r}\mathcal{Y}_r$ *for* $1 \leq r \leq n$.
*(2)* $\mathcal{Y}_r * \mathcal{Y}_l = p^{n-r}\mathcal{Y}_l = \mathcal{Y}_l * \mathcal{Y}_r$ *for* $1 \leq l \leq r \leq n$.
*(3)* $\mathcal{U}_0 * \mathcal{U}_0 = p^{n-1}(p-1)\mathcal{U}_0 + p^n \mathcal{Y}_1$.
*(4)* $\mathcal{U}_0 * \mathcal{Y}_r = p^{n-r}\mathcal{U}_0 = \mathcal{Y}_r * \mathcal{U}_0$ *for* $1 \leq r \leq n$.
*(5)* $\mathcal{U}_0 * (\mathcal{U}_0 - p^n I) * (\mathcal{U}_0 + p^{n-1} I) = 0$.

*The algebra* $H(K//K_0(p^n))$ *is an* $(n+1)$*-dimensional commutative algebra with generators* $\{\mathcal{U}_0, \mathcal{Y}_1, \ldots, \mathcal{Y}_n\}$ *and the above defining relations.*

We are interested in irreducible representations of $K$ having a $K_0(p^n)$-fixed vector. Let

$$I(n) := \operatorname{Ind}_{K_0(p^n)}^K 1 = \{\phi : K \to \mathbb{C} : \phi(k_0 k) = \phi(k) \text{ for } k_0 \in K_0,\ k \in K\}.$$

Then $I(n)$ is a right representation of $K$, denoted by $\pi_R$, where $\pi_R(k)\phi(k') = \phi(k'k)$, and $\dim I(n) = [K : K_0(p^n)] = p^{n-1}(p+1)$. By Frobenius reciprocity, every smooth irreducible representation of $K$ with a nonzero $K_0(p^n)$-fixed vector occurs in $I(n)$. Moreover, $I(n)$ is a sum of $n+1$ distinct irreducible representations and $I(n)^{K_0(p^n)} = H(K//K_0(p^n))$; see [1, Proposition 3.15].

We recall their explicit description. The algebra acts on $I(n)$ by

$$\pi_L(f)\phi(g) = \int_K f(k)\phi(k^{-1}g)\, dk \qquad (g \in K).$$

This action commutes with $\pi_R$, and on $I(n)^{K_0(p^n)}$ it is convolution: $\pi_L(f)\phi = f * \phi$. A basis of simultaneous eigenvectors is

$$v_1 = \mathcal{U}_0 + \mathcal{Y}_1, \qquad v_2 = \mathcal{U}_0 - p\mathcal{Y}_1, \qquad w_k = \mathcal{Y}_k - p\mathcal{Y}_{k+1} \quad (1 \leq k \leq n-1).$$

Their eigenvalues are given by the following table, in which $1 \leq r \leq n$ and $1 \leq k \leq n-1$:

| | $\mathcal{U}_0$ | $\mathcal{Y}_r$ |
|---|---|---|
| $v_1$ | $p^n$ | $p^{n-r}$ |
| $v_2$ | $-p^{n-1}$ | $p^{n-r}$ |
| $w_k$ | $0$ | $\begin{cases} 0, & r \leq k, \\ p^{n-r}, & r \geq k+1. \end{cases}$ |

For example, $\mathcal{U}_0 * v_1 = p^n v_1$ and $\mathcal{Y}_k * w_k = 0$. The latter corrects the entry in row $w_k$, column $\mathcal{Y}_k$, of [1, Table 1].

**Theorem 3** ([1], Corollary 3.18)**.** *The representation $I(n)$ is a sum of $n+1$ irreducible subspaces*

$$S_1 = \mathrm{Span}(\pi_R(K)v_1), \qquad S_2 = \mathrm{Span}(\pi_R(K)v_2), \qquad T_k = \mathrm{Span}(\pi_R(K)w_k)$$

*for $1 \le k \le n-1$, with $\dim S_1 = 1$, $\dim S_2 = p$ and $\dim T_k = p^{k-1}(p^2-1)$.*

For $i \ge 0$, let $\sigma(i)$ be the unique irreducible representation of $K$ with a $K_0(p^i)$-fixed vector and no $K_0(p^{i-1})$-fixed vector when $i \ge 1$; here $\sigma(0)$ is the trivial representation. Existence and uniqueness follow from [1, Corollary 3.16].

**Corollary 4.** *$S_1 \cong \sigma(0)$, $S_2 \cong \sigma(1)$ and $T_k \cong \sigma(k+1)$ for $1 \le k \le n-1$.*

*Proof.* The natural inclusion $I(0) \subseteq I(n)$ sends the constant function $1_K$ to $\mathcal{U}_0 + \mathcal{Y}_1 = v_1$, so $S_1 \cong \sigma(0)$. The $K_0(p)$-fixed line of the nontrivial constituent of $I(1)$ is spanned by $\mathcal{U}_0 - pI$. Its image in $I(n)$ is $v_2$, whose $K$-translates therefore span $\sigma(1)$. Similarly, $w_k$ is the image in $I(n)$ of the function $1_{K_0(p^k)} - p1_{K_0(p^{k+1})}$ belonging to $I(k+1)$. It is $K_0(p^{k+1})$-fixed and its average over $K_0(p^k)$ is zero. Each constituent of $I(k)$ has the same one-dimensional fixed line at levels $k$ and $k+1$, by Frobenius reciprocity and multiplicity one in $I(k+1)$. Hence $w_k$ lies in the new constituent $\sigma(k+1)$, and its $K$-translates span $T_k$. $\square$

The following observation justifies the multiplicities needed below.

**Lemma 5.** *Let $\pi$ have conductor exponent $c_\pi$. The multiplicity of $\sigma(j)$ in $\pi|_K$ is $0$ for $j < c_\pi$ and $1$ for $j \ge c_\pi$. Consequently, for $n \ge c_\pi$,*

$$V_\pi^{K_0(p^n)} \cong \bigoplus_{j=c_\pi}^{n} \sigma(j)^{K_0(p^n)}$$

*as $H(K//K_0(p^n))$-modules, with each summand one-dimensional.*

*Proof.* Let $a_j$ be the multiplicity of $\sigma(j)$ in $\pi|_K$. By Frobenius reciprocity and $I(m) \cong \bigoplus_{j=0}^{m} \sigma(j)$, the only $K$-types contributing $K_0(p^m)$-fixed vectors are these $\sigma(j)$, each with a one-dimensional fixed space. Thus, writing $d_m = \dim V_\pi^{K_0(p^m)}$ and $d_{-1} = 0$, we have

$$d_m = \sum_{j=0}^{m} a_j, \qquad a_m = d_m - d_{m-1} = \begin{cases} 0, & m < c_\pi, \\ 1, & m \ge c_\pi, \end{cases}$$

by Theorem 1. This proves the assertion. $\square$

Now we return to Tsuzuki's question and first illustrate the idea when $n = 2$. We seek an explicit $\phi \in H(K//K_0(p^2))$ whose trace is 1 for $\pi \in X_2^*$ and 0 for $\pi \in X_1^* \cup X_0^*$.

Let $\pi \in X_2^*$. By Lemma 5, its one-dimensional $K_0(p^2)$-fixed space comes from a $K$-subrepresentation $\rho_0 \cong T_1$. Let $\Theta : \rho_0 \longrightarrow T_1$ be a $K$-intertwining isomorphism and let $f_0 \neq 0$ be $K_0(p^2)$-fixed in $\rho_0$. Then $\Theta(f_0) = cw_1$ for some $c \neq 0$, and

$$\pi(\phi)f_0 = \Theta^{-1}\left(c\int_K \phi(g)\pi_R(g)w_1\,dg\right).$$

We claim that the integral on the right equals $\pi_L(\phi)w_1$. Indeed, put $\widehat{\phi}(g) = \phi(g^{-1})$. For $g \in K$ the double cosets $K_0(p^2)gK_0(p^2)$ and $K_0(p^2)g^{-1}K_0(p^2)$ coincide: this is checked on $1, w(1), y(p)$, using $w(1)^{-1} = -w(1)$ and conjugation by $\mathrm{diag}(-1,1)$ for $y(p)$. Thus $\widehat{\phi} = \phi$, and commutativity gives

$$\left(\int_K \phi(g)\pi_R(g)w_1\,dg\right)(k) = \int_K \phi(g)w_1(kg)\,dg = (w_1 * \widehat{\phi})(k) = (\phi * w_1)(k).$$

The same argument applies to every fixed line and every level $n$, using the representatives $1, w(1), y(p^r)$, $1 \leq r \leq n-1$. Hence choosing $\pi_L(\phi)w_1 = w_1$ gives $\pi(\phi)f_0 = f_0$.

Next assume $\pi \in X_1^*$. Its two-dimensional $K_0(p^2)$-fixed space is the sum of the fixed lines in copies of $S_2$ and $T_1$, by Lemma 5. To obtain trace zero, we therefore choose $\pi_L(\phi)w_1 = w_1$ and $\pi_L(\phi)v_2 = -v_2$. Finally, if $\pi \in X_0^*$, its three-dimensional fixed space comes from copies of $S_1, S_2, T_1$. We impose in addition $\pi_L(\phi)v_1 = 0$.

Using the table, it is easy to check that

$$\phi = \frac{1}{p+p^2}\mathcal{U}_0 - \frac{1+2p}{p+p^2}\mathcal{Y}_1 + \mathcal{Y}_2$$

has these eigenvalues, where $\mathcal{Y}_2 = I$. In bases of the corresponding fixed lines, the matrices of $\pi(\phi)$ are respectively

$$(1) \quad (\pi \in X_2^*), \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\pi \in X_1^*), \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\pi \in X_0^*).$$

For $n \geq 3$, the same construction assigns eigenvalue 1 to the fixed line of $\sigma(n)$, $-1$ to that of $\sigma(n-1)$ and 0 to those of the earlier types. By the table we can take $\phi = p^{-2}\mathcal{Y}_{n-2} - 2p^{-1}\mathcal{Y}_{n-1} + \mathcal{Y}_n$. For $n = 1$, the element $-2(p+1)^{-1}\mathcal{U}_0 + (p-1)(p+1)^{-1}\mathcal{Y}_1$ has eigenvalues $-1, 1$ on the fixed lines of $S_1, S_2$. For $n = 0$, we take $1_K$. We summarize the construction as follows.

**Proposition 6.** *For each fixed $n$, use Haar measure with* $\mathrm{vol}(K_0(p^n)) = 1$. *Define $\phi_n \in H(K//K_0(p^n))$ by*

$$\begin{aligned}
\phi_0 &= 1_K, \\
\phi_1 &= -\frac{2}{p+1}\mathcal{U}_0 + \frac{p-1}{p+1}\mathcal{Y}_1, \\
\phi_2 &= \frac{1}{p+p^2}\mathcal{U}_0 - \frac{1+2p}{p+p^2}\mathcal{Y}_1 + \mathcal{Y}_2, \\
\phi_n &= \frac{1}{p^2}\mathcal{Y}_{n-2} - \frac{2}{p}\mathcal{Y}_{n-1} + \mathcal{Y}_n \qquad (n \geq 3).
\end{aligned}$$

*Then, as an operator on $V_\pi^{K_0(p^n)}$,*

$$\mathrm{tr}\,\pi(\phi_n) = \begin{cases} 0, & \pi \in X_i^*, \quad i < n, \\ 1, & \pi \in X_n^*. \end{cases}$$

*Proof.* If $c_\pi = n$, only the fixed line of $\sigma(n)$ contributes, with eigenvalue 1. If $c_\pi < n$, Lemma 5 shows that both $\sigma(n-1)$ and $\sigma(n)$ contribute once, with eigenvalues $-1$ and 1, while all earlier types contribute zero. The case $n = 0$ follows from $\dim V_\pi^K = 1$ for $\pi \in X_0^*$. □

*Remark* 1 (Averaging operators and Casselman's formula). Fix $n$ and use its Haar measure throughout. Put $e_r = 1_{K_0(p^r)}/\,\mathrm{vol}(K_0(p^r))$ for $0 \leq r \leq n$, and $e_{-1} = e_{-2} = 0$. Then $\pi(e_r)$ is the averaging projection onto $V_\pi^{K_0(p^r)}$. The double-coset decomposition gives, for $n \geq 1$,

$$e_0 = \frac{\mathcal{U}_0 + \mathcal{Y}_1}{p^{n-1}(p+1)}, \qquad e_r = p^{r-n}\mathcal{Y}_r \quad (1 \leq r \leq n).$$

Consequently, all the above formulas can be written uniformly as

$$\phi_n = e_n - 2e_{n-1} + e_{n-2}.$$

The differences $E_0 = e_0$ and $E_j = e_j - e_{j-1}$ for $1 \leq j \leq n$ select the fixed lines of the respective types $\sigma(j)$. Thus $\phi_n = E_n - E_{n-1}$ for $n \geq 1$, explaining the eigenvalues $1, -1, 0$ used in the construction. Taking traces also gives

$$\mathrm{tr}\,\pi(\phi_n) = d_n - 2d_{n-1} + d_{n-2}, \qquad d_r = \dim V_\pi^{K_0(p^r)}, \quad d_{-1} = d_{-2} = 0.$$

Casselman's formula shows directly that this second difference is 1 at $c_\pi = n$ and 0 at lower conductor, in agreement with Proposition 6.

## References


[1] E. M. Baruch and S. Purkait, *Hecke algebras, new vectors and new forms on* $\Gamma_0(m)$, Math. Z. **287** (2017), no. 1–2, 705–733.

[2] W. Casselman, *On some results of Atkin and Lehner*, Math. Ann. **201** (1973), 301–314.