A Hecke algebra of level 4 and associated Whittaker functions

By Ehud Moshe BARUCH*) and Soma PURKAIT**)

(Communicated by Kenji FUKAYA, M.J.A., March 12, 2025)

Abstract: In [3], we gave a presentation of the PGL_2 Hecke algebra of level 4. We now use this presentation to describe the finite dimensional representations of the Hecke algebra and compute the Whittaker function for newforms of level 4.

Key words: Hecke algebras; Newforms; Whittaker functions.

Let $G = \mathrm{GL}_2(\mathbf{Q}_2)$ and $K_0(4)$ be the subgroup of G defined by

$$K_0(4) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbf{Z}_2) \ : \ c \in 4\mathbf{Z}_2 \right\}.$$

We compute the center of the Hecke algebra $H(G//K_0(4))$ and use it to describe its finite dimensional irreducible representations and compute the full Whittaker function of the new vector in the two supercuspidal representations of level 4.

1. Center of Hecke algebra $H(G//K_0(4))$. For $s \in \mathbf{Q}_2$ and $t \in \mathbf{Q}_2^\times$, we consider the following elements of G:

$$x(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad y(s) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix},$$
$$d(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}, \quad w(t) = \begin{pmatrix} 0 & -1 \\ t & 0 \end{pmatrix},$$
$$z(t) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}.$$

We recall the structure of the Hecke algebra $H(G//K_0(4))$, which is a double coset algebra of complex-valued locally constant compactly supported functions on G that are $K_0(4)$ bi-invariant with the usual convolution, as described in [3, Section 3.4].

By [3, Lemma 3.12], the algebra is supported on all the double coset representatives of G mod $K_0(4)$ (up to central elements $z(t)$):

$$d(2^n),\ w(2^n) \quad \text{for } n \in \mathbf{Z},$$
$$d(2^n)y(2) \quad \text{for } n \geq 0, \quad y(2)d(2^{-n}) \quad \text{for } n \geq 1,$$

and

$$w(2^n)y(2),\ y(2)w(2^n),\ y(2)w(2^n)y(2) \quad \text{for } n \geq 2.$$

Let $X_g \in H(G//K_0(4))$ be the characteristic function of $K_0(4)gK_0(4)$ and let

$$\text{for } n \in \mathbf{Z}, \qquad \mathcal{T}_n := X_{d(2^n)}, \quad \mathcal{U}_n := X_{w(2^n)},$$
$$\mathcal{V} := X_{y(2)}, \quad \mathfrak{Z} := X_{z(2)},$$

$$\text{for } n \geq 1, \qquad \mathcal{R}_n := \mathcal{T}_n * \mathcal{V}, \qquad \mathcal{S}_n := \mathcal{V} * \mathcal{T}_{-n},$$
$$\text{for } n \geq 2, \qquad \mathcal{W}_n := \mathcal{U}_n * \mathcal{V}, \qquad \mathcal{Y}_n := \mathcal{V} * \mathcal{U}_n,$$
$$\mathcal{Z}_n := \mathcal{V} * \mathcal{U}_n * \mathcal{V}.$$

Then, up to central elements $\mathfrak{Z}^n = X_{z(2^n)}$, the basis elements of $H(G//K_0(4))$ are precisely $\mathcal{T}_n, \mathcal{U}_n$ for $n \in \mathbf{Z}$, $\mathcal{V}$, $\mathcal{R}_n$, $\mathcal{S}_n$ for $n \geq 1$ and $\mathcal{W}_n$, $\mathcal{Y}_n$, $\mathcal{Z}_n$ for $n \geq 2$.

Theorem 1 ([3]). *The Hecke algebra*

$$H(G//K_0(4))/\langle \mathfrak{Z} \rangle$$

is generated by $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{V}$ *with the defining relations:*

$$\mathcal{U}_1^2 = 2(1+\mathcal{V}), \quad \mathcal{U}_2^2 = 1,$$
$$\mathcal{U}_1\mathcal{V} = \mathcal{V}\mathcal{U}_1 = \mathcal{U}_1, \quad \mathcal{U}_2\mathcal{V}\mathcal{U}_2 = \mathcal{V}\mathcal{U}_2\mathcal{V}.$$

The elements $\mathcal{T}_n$, $\mathcal{U}_n$ are obtained from $\mathcal{U}_1$, $\mathcal{U}_2$ ([3, Lemma 3.14]). Also we obtain $\mathcal{V}$ using the first relation above, so the generators $\mathcal{U}_1$, $\mathcal{U}_2$ are sufficient to describe the algebra. We also note that $\mathcal{V}^2 = 1$.

We will next describe the center of the above algebra. We need some more relations other than the ones noted in [3, Section 3.4]. These relations can be derived from the earlier relations ([3, Lemma 3.14, Proposition 3.15]) and observing equalities of certain double cosets, so we just state them. We also note that the relations below are modulo $\langle \mathfrak{Z} \rangle$.

2020 Mathematics Subject Classification. Primary 11F41; Secondary 11F70, 22E50.

*) Department of Mathematics, Technion-Israel Institute of Technology, Haifa, 32000, Israel.

**) School of Science, Institute of Science Tokyo, 2-12-1, Ookayama, Meguro-ku, Tokyo 152-8550, Japan.

doi: 10.3792/pjaa.101.004

Lemma 1.1. *For $n \geq 1$, $\mathcal{V}\mathcal{T}_n = \mathcal{T}_n$ and $\mathcal{T}_{-n}\mathcal{V} = \mathcal{T}_{-n}$ and consequently $\mathcal{V}\mathcal{R}_n = \mathcal{R}_n$, $\mathcal{S}_n\mathcal{V} = \mathcal{S}_n$. Further for $n \leq 1$, $\mathcal{U}_n = \mathcal{U}_n\mathcal{V} = \mathcal{V}\mathcal{U}_n = \mathcal{V}\mathcal{U}_n\mathcal{V}$.*

Lemma 1.2.

$$\begin{aligned}
\textit{For } n \leq 0,\quad & \mathcal{U}_1\mathcal{U}_n = 2\ \mathcal{T}_{-(1-n)} + 2\ \mathcal{S}_{1-n},\\
& \mathcal{U}_n\mathcal{U}_1 = 2\ \mathcal{T}_{1-n} + 2\ \mathcal{R}_{1-n}.\\
\textit{For } n \geq 1,\quad & \mathcal{U}_1\mathcal{T}_n = 2\ \mathcal{U}_{n+1} + 2\ \mathcal{Y}_{n+1},\\
& \mathcal{T}_{-n}\mathcal{U}_1 = 2\ \mathcal{U}_{n+1} + 2\ \mathcal{W}_{n+1},\\
& \mathcal{U}_1\mathcal{R}_n = 2\ \mathcal{W}_{n+1} + 2\ \mathcal{Z}_{n+1},\\
& \mathcal{R}_n\mathcal{U}_1 = \mathcal{U}_{1-n},\\
& \mathcal{U}_1\mathcal{S}_n = \mathcal{U}_{1-n},\\
& \mathcal{S}_n\mathcal{U}_1 = 2\ \mathcal{Y}_{n+1} + 2\ \mathcal{Z}_{n+1}.\\
\textit{For } n \geq 2,\quad & \mathcal{U}_1\mathcal{W}_n = \mathcal{R}_{n-1},\\
& \mathcal{W}_n\mathcal{U}_1 = \mathcal{T}_{1-n},\\
& \mathcal{U}_1\mathcal{Y}_n = \mathcal{T}_{n-1},\\
& \mathcal{Y}_n\mathcal{U}_1 = \mathcal{S}_{n-1},\\
& \mathcal{U}_1\mathcal{Z}_n = \mathcal{R}_{n-1},\\
& \mathcal{Z}_n\mathcal{U}_1 = \mathcal{S}_{n-1}.
\end{aligned}$$

Lemma 1.3.

$$\begin{aligned}
\textit{For } n \in \mathbf{Z},\quad & \mathcal{U}_2\mathcal{U}_n = \mathcal{T}_{n-2},\\
& \mathcal{U}_n\mathcal{U}_2 = \mathcal{T}_{2-n}.\\
\textit{For } n \in \mathbf{Z},\quad & \mathcal{U}_2\mathcal{T}_n = \mathcal{U}_{n+2},\\
& \mathcal{T}_n\mathcal{U}_2 = \mathcal{U}_{2-n}.\\
\textit{For } n \geq 1,\quad & \mathcal{U}_2\mathcal{R}_n = \mathcal{W}_{n+2},\\
& \mathcal{R}_n\mathcal{U}_2 = \mathcal{R}_n,\\
& \mathcal{U}_2\mathcal{S}_n = \mathcal{S}_n,\\
& \mathcal{S}_n\mathcal{U}_2 = \mathcal{Y}_{2+n}.\\
\textit{For } n = 2,\quad & \mathcal{U}_2\mathcal{Z}_2 = \mathcal{Y}_2,\\
& \mathcal{Z}_2\mathcal{U}_2 = \mathcal{W}_2.\\
\textit{For } n \geq 3,\quad & \mathcal{U}_2\mathcal{Z}_n = \mathcal{Z}_n = \mathcal{Z}_n\mathcal{U}_2.\\
\textit{For } n \geq 2,\quad & \mathcal{U}_2\mathcal{W}_n = \begin{cases} \mathcal{V} & \textit{if } n = 2\\ \mathcal{R}_{n-2} & \textit{if } n \geq 3,\end{cases}\\
& \mathcal{W}_n\mathcal{U}_2 = \begin{cases} \mathcal{Z}_2 & \textit{if } n = 2\\ \mathcal{W}_n & \textit{if } n \geq 3,\end{cases}\\
& \mathcal{U}_2\mathcal{Y}_n = \begin{cases} \mathcal{Z}_2 & \textit{if } n = 2\\ \mathcal{Y}_n & \textit{if } n \geq 3,\end{cases}\\
& \mathcal{Y}_n\mathcal{U}_2 = \begin{cases} \mathcal{V} & \textit{if } n = 2\\ \mathcal{S}_{n-2} & \textit{if } n \geq 3.\end{cases}
\end{aligned}$$

Theorem 2. *The center of the Hecke algebra $H(G//K_0(4))/\langle \mathfrak{Z} \rangle$ is non-trivial. In particular, any element of the center is of the form*

$$\begin{aligned}
Z = \alpha + \delta_2\mathcal{V}
&+ \sum_{n\geq 1}(2\delta_{n+4} - \delta_{n+2})(\mathcal{T}_n + \mathcal{T}_{-n} + \mathcal{R}_n + \mathcal{S}_n)\\
&+ \delta_3\mathcal{U}_1 + \sum_{n\leq 0}\delta_{4-n}\mathcal{U}_n + \sum_{n\geq 2}\delta_n(\mathcal{U}_n + \mathcal{Z}_n)\\
&+ \sum_{n\geq 2}(2\delta_{n+2} - \delta_n)(\mathcal{W}_n + \mathcal{Y}_n),
\end{aligned}$$

where α and δ_n, $n \geq 2$ are any scalars. Thus the center has at least three linearly independent elements given by 1, $\mathcal{V} + \mathcal{U}_2 + \mathcal{Z}_2 - \mathcal{W}_2 - \mathcal{Y}_2$ and $(\mathcal{T}_1 + \mathcal{T}_{-1} + \mathcal{R}_1 + \mathcal{S}_1) - \mathcal{U}_1 - (\mathcal{U}_3 + \mathcal{Z}_3) + (\mathcal{W}_3 + \mathcal{Y}_3)$.

Proof. Let

$$\begin{aligned}
Z = \sum_{n\geq 0}\alpha_n\mathcal{T}_n + \sum_{n\geq 1}\beta_n\mathcal{T}_{-n} + \sum_{n\leq 1}\gamma_n\mathcal{U}_n
&+ \sum_{n\geq 2}\delta_n\mathcal{U}_n + \lambda_0\mathcal{V} + \sum_{n\geq 1}\lambda_n\mathcal{R}_n\\
&+ \sum_{n\geq 1}\sigma_n\mathcal{S}_n + \sum_{n\geq 2}\eta_n\mathcal{W}_n + \sum_{n\geq 2}\theta_n\mathcal{Y}_n\\
&+ \sum_{n\geq 2}\kappa_n\mathcal{Z}_n
\end{aligned}$$

be an element of the center of

$$H(\mathrm{GL}_2(\mathbf{Q}_2)//K_0(4))/\langle \mathfrak{Z} \rangle.$$

Since Z commutes with $\mathcal{V}$, it follows by Lemma 1.1 that

$$\begin{aligned}
&\alpha_n = \lambda_n,\quad \beta_n = \sigma_n \quad \text{for } n \geq 1,\\
&\delta_n = \kappa_n,\quad \eta_n = \theta_n \quad \text{for } n \geq 2.
\end{aligned}$$

Next we use Lemma 1.2 along with Z commuting with $\mathcal{U}_1$ to further get that

$$\alpha_n = \beta_n \text{ for } n \geq 1, \qquad \delta_n + \theta_n = 2\gamma_{2-n} \text{ for } n \geq 2.$$

Finally we use Lemma 1.3 and commuting relation of Z with $\mathcal{U}_2$ to get the following conditions:

$$\begin{aligned}
&\lambda_0 = \delta_2, \quad \text{for } n \geq 2,\ \delta_n + \theta_n = 2\gamma_{2-n},\\
&\text{for } n \geq 3,\ \alpha_{n-2} = \theta_n, \quad \delta_n = \gamma_{4-n}.
\end{aligned}$$

Note that the first and fourth relations above imply that $\delta_3 = \gamma_1$ and $\delta_n + \theta_n = 2\delta_{n+2}$ for $n \geq 2$. Using these relations we get the general form of an element in the center.

Note that if $\delta_n = 0$ for all n, we get the trivial element 1. If $\alpha_0 = 0$ and $\delta_2 = 1$ and $\delta_n = 0$ for

$n \geq 3$ we get the element $\mathcal{V} + \mathcal{U}_2 + \mathcal{Z}_2 - \mathcal{W}_2 - \mathcal{Y}_2$ and with the choice of scalars $\alpha_0 = 0$, $\delta_2 = 0$, $\delta_3 = 1$ and $\delta_n = 0$ for $n \geq 4$ we get the element $(\mathcal{T}_1 + \mathcal{T}_{-1} + \mathcal{R}_1 + \mathcal{S}_1) - \mathcal{U}_1 - (\mathcal{U}_3 + \mathcal{Z}_3) + (\mathcal{W}_3 + \mathcal{Y}_3)$. □

The following proposition can be easily verified using the relations:

Proposition 1.4. *Let $C_1 := \mathcal{V} + \mathcal{U}_2 + \mathcal{Z}_2 - \mathcal{W}_2 - \mathcal{Y}_2$. Then*

$$(C_1 + 5)(C_1 - 1) = 0.$$

In particular, C_1 is an invertible central element.

The central elements $(mI + nC_1)$ where $(m, n) = (\frac{1}{6}, \frac{-1}{6}), (\frac{5}{6}, \frac{1}{6})$ are non-trivial idempotents in the Hecke algebra.

We conjecture that the three linearly independent elements 1, $C_1 = \mathcal{V} + \mathcal{U}_2 + \mathcal{Z}_2 - \mathcal{W}_2 - \mathcal{Y}_2$ and $C_2 := (\mathcal{T}_1 + \mathcal{T}_{-1} + \mathcal{R}_1 + \mathcal{S}_1) - \mathcal{U}_1 - (\mathcal{U}_3 + \mathcal{Z}_3) + (\mathcal{W}_3 + \mathcal{Y}_3)$ generate the center of the Hecke algebra.

Remark. In [3], we proved the local Shimura correspondence, i.e., the Hecke algebras of metaplectic group $\widetilde{\mathrm{SL}_2}(\mathbf{Q}_2)$ of level 8 with certain quadratic characters are isomorphic to $H(\mathrm{GL}_2(\mathbf{Q}_2)//K_0(4))/\langle \mathfrak{Z} \rangle$. Thus Theorem 2 also gives a description of the center of $H(\widetilde{\mathrm{SL}_2}(\mathbf{Q}_2)//\overline{K_0(8)}, \chi_1)$.

2. Finite dimensional representations of $\boldsymbol{H := H(G//K_0(4))/\langle \mathfrak{Z} \rangle}$. We will from now on normalize the element $\mathcal{U}_1$ by $\mathcal{U}_1/\sqrt{2}$ and will use the same notation $\mathcal{U}_1$ for it. We will also normalise the central element C_2 by $C_2/\sqrt{2}$ and by abuse of notation denote the normalised element by C_2 itself. Thus, we have the following generating relations for the Hecke algebra:

$$\mathcal{U}_1^2 = 1 + V, \quad \mathcal{V}^2 = 1, \quad \mathcal{U}_2^2 = 1,$$
$$\mathcal{U}_1\mathcal{V} = \mathcal{V}\mathcal{U}_1 = \mathcal{U}_1, \quad \mathcal{U}_2\mathcal{V}\mathcal{U}_2 = \mathcal{V}\mathcal{U}_2\mathcal{V}.$$

In particular, $\mathcal{U}_1^3 = 2\mathcal{U}_1$, and the two central elements in this algebra are given by

$$\begin{aligned} C_1 &= \mathcal{V} + \mathcal{U}_2 + \mathcal{V}\mathcal{U}_2\mathcal{V} - \mathcal{U}_2\mathcal{V} - \mathcal{V}\mathcal{U}_2, \\ C_2 &= \mathcal{U}_1\mathcal{U}_2 + \mathcal{U}_2\mathcal{U}_1 + \mathcal{U}_1\mathcal{U}_2\mathcal{V} + \mathcal{V}\mathcal{U}_2\mathcal{U}_1 - \mathcal{U}_1 \\ &\quad - \mathcal{U}_2\mathcal{U}_1\mathcal{U}_2 - \mathcal{V}\mathcal{U}_2\mathcal{U}_1\mathcal{U}_2\mathcal{V} + \mathcal{U}_2\mathcal{U}_1\mathcal{U}_2\mathcal{V} \\ &\quad + \mathcal{V}\mathcal{U}_2\mathcal{U}_1\mathcal{U}_2. \end{aligned}$$

Let V be an irreducible finite dimensional representation of H such that $\dim(V) > 1$. We first assume that $\mathcal{U}_1$ has a zero eigenvalue in V. That is, there exists non-zero $\phi \in V$ such that $\mathcal{U}_1\phi = 0$. Thus, $(1 + \mathcal{V})\phi = \mathcal{U}_1^2\phi = 0$, and so $\mathcal{V}\phi = -\phi$. If ϕ is an eigenvector for $\mathcal{U}_2$, then it is a common eigenvector of H and hence the subspace spanned by ϕ is a proper invariant subspace contradicting the irreducibility of V. Define $\psi := \mathcal{U}_2\phi$. Then ψ, ϕ are linearly independent and from $\mathcal{U}_2^2 = 1$ we get $\mathcal{U}_2(\psi) = \phi$.

Now,

$$\mathcal{U}_2\mathcal{V}\mathcal{U}_2(\phi) = \mathcal{U}_2\mathcal{V}\psi, \quad \mathcal{V}\mathcal{U}_2\mathcal{V}\phi = -\mathcal{V}\mathcal{U}_2\phi = -\mathcal{V}\psi$$

and so from $\mathcal{U}_2\mathcal{V}\mathcal{U}_2 = \mathcal{V}\mathcal{U}_2\mathcal{V}$ we get that

$$\mathcal{U}_2\mathcal{V}\psi = -\mathcal{V}\psi. \tag{1}$$

To determine $\mathcal{V}\psi$ we notice that since C_1 is in the center and since V is irreducible, there exists $\lambda \in \mathbf{C}$ such that $C_1 v = \lambda v$ for all $v \in V$. In particular,

$$\begin{aligned} \lambda\phi = C_1\phi &= (\mathcal{V} + \mathcal{U}_2 + \mathcal{V}\mathcal{U}_2\mathcal{V} - \mathcal{U}_2\mathcal{V} - \mathcal{V}\mathcal{U}_2)\phi \\ &= -\phi + \psi - \mathcal{V}\psi + \psi - \mathcal{V}\psi, \end{aligned}$$

and so

(2)
$$\mathcal{V}\psi = \psi - \frac{1+\lambda}{2}\phi, \quad \text{i.e.,} \quad \mathcal{U}_2\mathcal{V}\psi = \phi - \frac{1+\lambda}{2}\psi.$$

Using (1) and (2) together with the fact that ϕ and ψ are linearly independent it follows that $\lambda = 1$ and

$$\mathcal{V}\psi = \psi - \phi.$$

We now have two possibilities. Either $\mathcal{U}_1\psi$ is in the space spanned by ϕ and ψ or $\mathcal{U}_1\psi$, ϕ and ψ are linearly independent. We shall see that both cases are possible with the first one leading to two particular 2-dimensional irreducible representations of H and the second case leading to a one parameter family of 3-dimensional irreducible representations of H.

2.1. 2-dimensional irreducible representations. We assume that there exist scalars $a, b \in \mathbf{C}$ such that $\mathcal{U}_1\psi = a\psi + b\phi$. Since $\mathcal{U}_1\phi = 0$, we have $\mathcal{U}_1^2\psi = a^2\psi + ab\phi = (1 + \mathcal{V})\psi = 2\psi - \phi$. Hence $a = \epsilon\sqrt{2}$ and $b = -\epsilon\frac{\sqrt{2}}{2}$ with $\epsilon = \pm 1$. It is easy to show that each ϵ gives a 2-dimensional irreducible representation of H, and a different choice of ϵ gives a non-isomorphic representation. We denote the representation with $\epsilon = 1$ by V^+ and the representation with $\epsilon = -1$ by V^-. We will later show that these are the only 2-dimensional irreducible representations of H.

2.2. 3-dimensional irreducible representations. We now assume that $\mathcal{U}_1\psi$ is not in the space spanned by ϕ and ψ and define $\eta := \mathcal{U}_1\psi$. Then

$$\begin{aligned} \mathcal{U}_1^2\psi &= \mathcal{U}_1\eta = (1 + \mathcal{V})\psi = 2\psi - \phi, \\ \mathcal{U}_1^2\eta &= \mathcal{U}_1(2\psi - \phi) = 2\eta = (1 + \mathcal{V})\eta = \eta + \mathcal{V}\eta. \end{aligned}$$

Hence $\mathcal{V}\eta = \eta$. It remains to find $\mathcal{U}_2\eta$. To do that we consider the second central element C_2. There exists a scalar λ such that $C_2\phi = \lambda\phi$. Hence

$$\begin{aligned}\lambda\phi &= (\mathcal{U}_1\mathcal{U}_2 + \mathcal{U}_2\mathcal{U}_1 + \mathcal{U}_1\mathcal{U}_2\mathcal{V} + \mathcal{V}\mathcal{U}_2\mathcal{U}_1 - \mathcal{U}_1 \\ &\quad - \mathcal{U}_2\mathcal{U}_1\mathcal{U}_2 - \mathcal{V}\mathcal{U}_2\mathcal{U}_1\mathcal{U}_2\mathcal{V} \\ &\quad + \mathcal{U}_2\mathcal{U}_1\mathcal{U}_2\mathcal{V} + \mathcal{V}\mathcal{U}_2\mathcal{U}_1\mathcal{U}_2)\phi \\ &= \eta + 0 - \eta + 0 - 0 - \mathcal{U}_2\eta + \mathcal{V}\mathcal{U}_2\eta - \mathcal{U}_2\eta \\ &\quad + \mathcal{V}\mathcal{U}_2\eta.\end{aligned}$$

Applying $\mathcal{U}_2$ to the above equation we get

$$\begin{aligned}\lambda\psi = \mathcal{U}_2(\lambda\phi) &= -2\mathcal{U}_2^2\eta + 2\mathcal{U}_2\mathcal{V}\mathcal{U}_2\eta \\ &= -2\eta + 2\mathcal{V}\mathcal{U}_2\mathcal{V}\eta \\ &= -2\eta + 2\mathcal{V}\mathcal{U}_2\eta.\end{aligned}$$

Now applying $\mathcal{V}$ to the above relation we get that $\mathcal{U}_2\eta = \eta + \frac{\lambda}{2}\psi - \frac{\lambda}{2}\phi$. It is easy to check that the space $V = V(\lambda) = \operatorname{span}\{\phi, \psi, \eta\}$ with the following action of H is a representation of H:

$$\begin{aligned}&\mathcal{U}_1\phi = 0, \quad \mathcal{U}_1\psi = \eta, \quad \mathcal{U}_1\eta = 2\psi - \phi, \\ &\mathcal{V}\phi = -\phi, \quad \mathcal{V}\psi = \psi - \phi, \quad \mathcal{V}\eta = \eta, \\ &\mathcal{U}_2\phi = \psi, \quad \mathcal{U}_2\psi = \phi, \quad \mathcal{U}_2\eta = \eta + \frac{\lambda}{2}\psi - \frac{\lambda}{2}\phi.\end{aligned}$$

We now check when $V(\lambda)$ is irreducible. The eigenvectors of $\mathcal{U}_1$ in $V(\lambda)$ are

$$\phi, \quad \phi - 2\psi - \sqrt{2}\eta, \quad \phi - 2\psi + \sqrt{2}\eta$$

with eigenvalues $0, \sqrt{2}, -\sqrt{2}$ respectively. A 1-dimensional invariant subspace of $V(\lambda)$ must be spanned by one of these vectors. It cannot be spanned by ϕ since ϕ is not an eigenvector of $\mathcal{U}_2$. The other two vectors are eigenvectors of $\mathcal{V}$ corresponding to eigenvalue 1. Further they are eigenvectors of $\mathcal{U}_2$ with eigenvalue 1 precisely when $\lambda = 3\sqrt{2}$ and $\lambda = -3\sqrt{2}$. We now look for 2-dimensional invariant subspaces of $V(\lambda)$. Since $\mathcal{U}_1$ is diagonalizable, such a subspace has to be spanned by two eigenvectors of $\mathcal{U}_1$. It is easy to check that $\mathcal{U}_2$ does not preserve these subspaces and hence there are no 2-dimensional invariant subspaces in $V(\lambda)$. Hence we have proved the following theorem:

Theorem 3. *Let V be a finite dimensional irreducible representation of H with $\dim(V) > 1$ such that V contains an eigenvector of $\mathcal{U}_1$ with eigenvalue 0. Then V is isomorphic to exactly one of V^+, V^- or $V(\lambda)$ with $\lambda \neq \pm 3\sqrt{2}$, where $\dim(V^+) = \dim(V^-) = 2$ and $\dim(V(\lambda)) = 3$. Moreover, $V(\lambda)$ is irreducible if and only if $\lambda \neq \pm 3\sqrt{2}$.*

Theorem 4. *Let V be a finite dimensional irreducible representation of H with $\dim(V) > 1$. Then V contains an eigenvector of $\mathcal{U}_1$ with eigenvalue 0.*

Proof. The proof is by contradiction and uses similar arguments as in Theorem 3. We skip the details. □

Note that the above two theorems give a complete classification of the irreducible representations of H.

3. Whittaker functions for new vectors of level 4. We now determine the complete 2-adic Whittaker function for a new vector of level 4. By Atkin and Li [1, Theorem 4.4 (iv) and Remark on p. 236], we know that the 2-adic representation appearing in the automorphic representation of a new form of level $4M$ where M is odd is supercuspidal. We now prove that there are exactly two such supercuspidal representations the choice of which depends on the sign of the Atkin-Lehner involution on the form.

If $f \in S_{2k}(\Gamma_0(4M))$ is a new form and an eigenform for all Hecke operators T_p where p is prime to $4M$, then we have $U_2(f) = 0$ and $W_4(f) = \epsilon f$ where U_2 and W_4 are the Atkin-Lehner operator and involution respectively [1]. If ϕ_f is the corresponding automorphic form on $\mathrm{GL}_2(\mathbf{A_Q})$, it follows from [2, Proposition 4.1 and Proposition 5.11] that $\mathcal{V}\phi_f = -\phi_f$ and $\mathcal{U}_2\phi_f = \epsilon\phi_f$. Since $\mathcal{U}_1^2 = (1 + \mathcal{V})$ and $\mathcal{U}_1^3 = 2\mathcal{U}_1$, it follows that $\mathcal{U}_1\phi_f = 0$. Hence, ϕ_f is an eigenform of the Hecke operators $\mathcal{U}_1$ and $\mathcal{U}_2$ with eigenvalues 0 and ϵ respectively.

We let $W_f = W_{\phi_f}$ be the Whittaker function attached to f and $W_{f,2}$ be the 2-adic component of this Whittaker function when decomposed into local Whittaker functions. This function is uniquely determined up to a constant and it is right invariant by $K_0(4)$.

Let ψ be the usual unramified additive character of $\mathbf{Q}_2$ such that $\psi(t) = -1$ if and only if $|t| = 2$. Also let $\mathcal{W}(G, \psi)^{K_0(4)}$ be the space of Whittaker functions which are right invariant by $K_0(4)$. The Hecke algebra H acts on $\mathcal{W}(G, \psi)^{K_0(4)}$ in the following way: for $\mathcal{T} \in H$ and $W \in \mathcal{W}(G, \psi)^{K_0(4)}$,

$$\mathcal{T}(W)(g) = \int_G W(gh)\mathcal{T}(h)dh,$$

where dh is a Haar measure on G normalized so that $K_0(4)$ has measure one. It follows that $W_{f,2}$ satisfies

$$\mathcal{U}_1(W_{f,2}) = 0, \quad \mathcal{U}_2(W_{f,2}) = \epsilon W_{f,2}$$

where $\epsilon = \pm 1$. We now show that these relations completely determine $W_{f,2}$.

Theorem 5. *Let* $W \in \mathcal{W}(G,\psi)^{K_0(4)}$ *be nonzero and such that*

$$W(zg) = W(g) \quad \textit{for } z \in Z, g \in G$$

and that

$$\mathcal{U}_1(W) = 0, \quad \mathcal{U}_2(W) = \epsilon W.$$

Then, up to a normalization by a scalar, W is given by

$$W(g) = \begin{cases} 1 & \textit{if } g = d(t) \textit{ with } |t| = 1 \\ -1 & \textit{if } g = d(t)y(2) \textit{ with } |t| = 1 \\ \epsilon & \textit{if } g = d(t)w(1) \textit{ with } |t| = 4 \\ 0 & \textit{if } g = d(t) \textit{ with } |t| \neq 1 \textit{ or} \\ & g = d(t)y(2) \textit{ with } |t| \neq 1 \textit{ or} \\ & g = d(t)w(1) \textit{ with } |t| \neq 4. \end{cases}$$

Proof. We first compute $\mathcal{U}_1(W)$. Using double coset decompositions in [2, Lemma 3.13], for $g \in G$, we have

$$\begin{aligned} \mathcal{U}_1(W)(g) &= \int_{K_0(4)w(2)K_0(4)} W(gh)dh \\ &= W(gw(2)) + W(gx(1)w(2)) = 0. \end{aligned}$$

Hence for all $g \in G$,

$$W(gw(2)) = -W(gx(1)w(2)).$$

In particular for $g = d(t)$ we get

$$\begin{aligned} W(d(t)w(2)) &= -W(d(t)x(1)w(2)) \\ &= -W(x(t)d(t)w(2)) \\ &= -\psi(t)W(d(t)w(2)). \end{aligned}$$

Since $d(t)w(2) = d(t/2)w(1)z(2)$, we get that

$$W(d(t/2)w(1)) = -\psi(t)W(d(t/2)w(1)).$$

Since $\psi(t) = -1$ if and only if $|t| = 2$, we get that $W(d(s)w(1)) = 0$ if $|s| \neq 4$.

Taking $g = d(t)(w(2))^{-1}$ we get that

$$W(d(t)) = -W(d(t)y(-2)) = -W(d(-t)y(2)).$$

for all $t \in \mathbf{Q}_2^*$.

We now apply $\mathcal{U}_2$ to W. For all $g \in G$, we have

$$\begin{aligned} \mathcal{U}_2(W)(g) &= \int_{K_0(4)w(4)K_0(4)} W(gh)dh \\ &= \int_{w(4)K_0(4)} W(gh)dh = W(gw(4)) \\ &= \epsilon W(g). \end{aligned}$$

For $g = d(t)$ we have

$$W(d(t)w(4)) = \epsilon W(d(t)).$$

Since $d(t)w(4) = d(t/4)w(1)z(4)$, we get that $W(d(t)) = 0$ if $|t| \neq 1$. If $|t| = 1$, then $d(t) \in K_0(4)$ and so $W(d(t)) = W(d(1)) = W(1)$. In particular, we also get that if $|t| = 4$, then $W(d(t)w(1)) = \epsilon W(1)$.

Using the Iwasawa decomposition $G = BK$, where B is the subgroup of upper triangular matrices and K the maximal compact, and [2, Lemma 3.6],

$$G = BK_0(4) \bigcup Bw(1)K_0(4) \bigcup By(2)K_0(4).$$

It now follows that W is identically zero if $W(1) = 0$ and that the formula in the theorem determines W completely. $\square$

References

[1] A. O. L. Atkin and W.-C. W. Li, Twists of newforms and pseudo-eigenvalues of W-operators, Invent. Math. **48** (1978), no. 3, 221–243.

[2] E. M. Baruch and S. Purkait, Hecke algebras, new vectors and new forms on $\Gamma_0(m)$, Math. Z. **287** (2017), no. 1–2, 705–733.

[3] E. M. Baruch and S. Purkait, Newforms of half-integral weight: the minus space of $S_{k+1/2}(\Gamma_0(8M))$, Israel J. Math. **232** (2019), no. 1, 41–73.