HECKE SUBALGEBRAS AND LOCAL NEWFORMS FOR THE METAPLECTIC DOUBLE COVER OF $\mathrm{SL}_2(\mathbb{Q}_p)$

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ABSTRACT. Let p be an odd prime, and let $\overline{\mathrm{SL}_2(\mathbb{Z}_p)}$ and $\overline{K_0(p^n)}$ denote the inverse images of $\mathrm{SL}_2(\mathbb{Z}_p)$ and the congruence subgroup $K_0(p^n)$ in the metaplectic double cover $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$. For $n \geq 2$, we study the subalgebra of the genuine Hecke algebra $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)//\overline{K_0(p^n)}, \eta)$ consisting of functions supported in $\overline{\mathrm{SL}_2(\mathbb{Z}_p)}$, where η is the genuine extension of either the trivial or the nontrivial quadratic character modulo p. We give an explicit basis and a presentation by generators and relations, and prove that this subalgebra is commutative of dimension $2n$. We give the multiplicity-free decomposition of $\mathrm{Ind}_{\overline{K_0(p^n)}}^{\overline{\mathrm{SL}_2(\mathbb{Z}_p)}} \eta$, determining the dimensions of its irreducible constituents and their associated Hecke eigenvalues.

This Hecke subalgebra acts on the $(\overline{K_0(p^n)}, \eta)$-fixed spaces in irreducible admissible genuine representations of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$. We compute its action explicitly on newvectors of prescribed η-type in principal series, Steinberg, even Weil, and supercuspidal representations. For supercuspidal representations, we construct the vectors by compact induction and use Ishimoto's conductor and dimension formulas.

Finally, we relate our operator $\mathcal{W}_{n-1}$ to Ishimoto's local realization of Ueda's twisting operator.

1. INTRODUCTION

The theory of modular forms of half-integral weight is closely tied to the representation theory of the metaplectic double cover of SL_2. Shimura's correspondence provides the basic bridge from half-integral to integral weight forms [6], while Kohnen's plus space and newform theory isolate distinguished Hecke-stable subspaces on the half-integral-weight side [4, 5]. Ueda extended this theory to nonsquarefree odd level by introducing twisting operators and a corresponding decomposition of the Kohnen space [7, 8]. From the local point of view, these spaces should be reflected in genuine representations and genuine Hecke algebras of the metaplectic group. This makes it natural to seek operator-theoretic descriptions of newforms, rather than relying only on orthogonal complements.

A useful model for this approach is the integral-weight work of Baruch and Purkait [1]. There, generators and relations for local Hecke algebras on $\mathrm{GL}_2(\mathbb{Z}_p)$ are used to characterize local newvectors and, after passage to classical operators, global newform spaces. In half-integral weight, the same philosophy was developed in [2, 3]. The paper [2] studies genuine Iwahori-type

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Hecke algebras for the double cover of $\mathrm{SL}_2(\mathbb{Q}_p)$ and uses their operators to characterize the minus-space counterpart to Kohnen's plus space at level $4M$. The theory at level $8M$ in [3] is built from the corresponding 2-adic Hecke algebras and gives a further minus-space decomposition compatible with the relevant Shimura–Ueda correspondence. The representation-theoretic interpretation of the 2-adic theory is also closely related to the work of Loke and Savin [14]; more generally, unramified genuine representations of covering groups were studied by Savin [22].

Local newform theory for metaplectic representations has developed in parallel. Roberts and Schmidt introduced local oldforms and newforms for genuine representations of the metaplectic group and related the total number of newforms to the available Whittaker models [20]. For SL_2, the conductor and newform theory of Lansky and Raghuram [13] is particularly useful for the linear representations underlying compactly induced metaplectic representations. Manderscheid's classification of genuine supercuspidal representations [11, 12], together with the Kutzko–Sally description on the linear side [10], supplies the compact types used in the supercuspidal case. Ishimoto has given explicit conductor formulas and formulas for the dimensions of the corresponding fixed-type spaces in irreducible genuine representations of the rank-one metaplectic group, and studied their relation with the local theta correspondence [16].

The main purpose of the present paper is to continue the Hecke algebra approach of [1, 2, 3] at higher odd-prime depth. Let p be odd, $K = \mathrm{SL}_2(\mathbb{Z}_p)$, and $K_0(p^n)$ the usual congruence subgroup, with $n \geq 2$. For a subgroup $H \subseteq \mathrm{SL}_2(\mathbb{Q}_p)$, write $\overline{H}$ for its full inverse image in the metaplectic double cover. For η the genuine extension to $\overline{K_0(p^n)}$ of either the trivial or the nontrivial quadratic character modulo p, we determine explicitly the subalgebra

$$H(\overline{K}//\overline{K_0(p^n)}, \eta)$$

of the genuine Hecke algebra consisting of functions supported in $\overline{K}$. We determine its supporting double cosets, generators, relations, and characters, and prove that it is commutative of dimension $2n$. In particular, the operators $\mathcal{U}_0$, $\mathcal{V}_{r,1}$ and $\mathcal{V}_{r,\lambda}$, together with the combinations $\mathcal{Y}_r$ and $\mathcal{W}_r$, make the structure of the algebra explicit. We determine the multiplicity-free decomposition of $\mathrm{Ind}_{\overline{K_0(p^n)}}^{\overline{K}}\eta$, including the dimensions and Hecke eigenvalues of its irreducible constituents. The Hecke algebra calculations forming the core of this paper were undertaken independently of Ishimoto's work and arise directly from our earlier operator-theoretic approach.

We work throughout with a fixed quadratic type η, and hence with the η-conductor

$$c_\eta(\pi) = \min\{n : V_\eta^{K_0(p^n)}(\pi) \neq 0\},$$

rather than the minimum of $c_\eta(\pi)$ over all admissible characters. This is the conductor naturally seen by the Hecke subalgebra associated with the fixed character η. It is also the appropriate notion for the intended global application: under adelization, the character η is induced by the local p-component of the nebentypus.

The Hecke subalgebra acts on the $(\overline{K_0(p^n)}, \eta)$-fixed spaces in irreducible admissible genuine representations of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$. We use Ishimoto's conductor and dimension formulas [16] in our study of the Hecke action on newvectors of prescribed η-type. We compute the action directly on explicit vectors in these spaces. For induced representations we write down the relevant isotypic vectors and compute their Hecke eigenvalues. The reducible principal-series cases lead to the even Weil and Steinberg representations, where the exact sequence relating these representations gives the dimensions of the type spaces and the corresponding operator actions.

For supercuspidal representations, we use Manderscheid's compact-induction classification [11, 12] in the explicit form recalled by Ishimoto, together with his description of the conductor and defect of the inducing types and his conductor and dimension formulas [16]. Our construction of newvectors adapts the ideas of Lansky–Raghuram [13, §3.3] to the metaplectic setting. We then compute the Hecke action directly on these vectors by reducing it to explicit operators on the underlying linear inducing types. This makes explicit the distinction among the unramified packets of cardinality two and four and the ramified packets.

While this paper was being completed, Ishimoto informed us of his manuscript *Kohnen–Ueda local newforms* [17]. His construction is complementary to ours. Ishimoto defines the operators U_{ϖ^2} and R_{ϖ} on the tower of η-type spaces and uses their kernels and images, together with level-changing maps, to define Kohnen–Ueda old and new quotients. His supercuspidal calculations use a transfer to the Kirillov model. By contrast, we study a finite-dimensional commutative Hecke subalgebra acting within a fixed $(\overline{K_0(p^n)}, \eta)$-type, calculate its presentation and characters, and act with its operators directly on the vectors under consideration; in the supercuspidal case our construction remains in the compact-induction model.

The relation with Ueda's classical theory is nevertheless quite concrete. Ishimoto's R_p is the local unipotent average corresponding, after normalization, to Ueda's quadratic twisting operator. In Section 5 we show that its restriction to a fixed $(\overline{K_0(p^n)}, \eta)$-type, followed by a lifted GL_2-conjugation, gives the action of $\mathcal{W}_{n-1}$ up to an explicit scalar; when n is odd, the conjugation also interchanges the two quadratic types. This is a comparison of fixed-type actions, not an equality of the original operators, whose supports and natural domains are different.

The classical interpretation is one of the principal motivations for the present local calculations, but we do not pursue the global theory here. In subsequent work, we plan to formulate and test a global oldform–newform characterization, beginning with levels $4p^n$ and $8p^n$. At the prime 2, we would use the established Kohnen plus-space operator in the first case and the 2-adic Hecke operators developed in our earlier work at level $8M$ in the second. At the odd prime p, the classical counterparts of the operators from our Hecke subalgebra should be compared with Ueda's twisting decomposition for the prescribed local nebentypus. Although we work over $\mathbb{Q}_p$ in order to keep this classical application and the normalizations explicit, the local constructions are expected to extend, with the appropriate changes of notation and normalization, to non-archimedean local fields of odd residual characteristic.

The paper is organized as follows. Section 2 introduces the Hecke subalgebra supported in $\overline{K}$ and determines its double-coset support. Section 3 computes the relations among its generators and the resulting characters and $\overline{K}$-types. Section 4 applies the algebra to local newforms, first for induced representations, then for special representations, and finally for compactly induced supercuspidal representations. Section 5 explains the relation among Ueda's twisting operator, Ishimoto's local operator R_p, and the operator $\mathcal{W}_{n-1}$ of this paper.

2. A Hecke Subalgebra of $\widetilde{G}$ modulo $\overline{K_0(p^n)}$

Let p be a prime and put $G = \mathrm{SL}_2(\mathbb{Q}_p)$. We write $\widetilde{G} = \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ for the non-trivial central extension of G by $\mu_2 = \{\pm 1\}$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p) & \longrightarrow & \mathrm{SL}_2(\mathbb{Q}_p) & \longrightarrow & 1 \\ & & \{(I, \pm 1)\} & & (g, \pm 1) & \longmapsto & g & & \end{array}$$

with a group law determined by the following 2-cocycle. Let $(\cdot,\cdot)_p$ be the Hilbert symbol over $\mathbb{Q}_p$. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$, define

$$\tau(A) = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{if } c = 0 \end{cases};$$

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p)$, set

$$s_p(g) = \begin{cases} (c,d)_p & \text{if } cd \neq 0 \text{ and } \mathrm{ord}_p(c) \text{ is odd} \\ 1 & \text{else.} \end{cases}$$

For $g, h \in G$, define

$$\sigma_0(g,h) = \left(\frac{\tau(gh)}{\tau(g)}, \frac{\tau(gh)}{\tau(h)} \right)_p = \big(\tau(gh)\tau(g), \tau(gh)\tau(h)\big)_p.$$

Define the 2-cocycle:

$$c(g,h) = \sigma_0(g,h) s_p(g) s_p(h) s_p(gh). \tag{1}$$

Thus $\widetilde{G}$ is the set $\mathrm{SL}_2(\mathbb{Q}_p) \times \mu_2$ with the group law given by

$$(g,\ \epsilon_1)(h,\ \epsilon_2) = (gh,\ \epsilon_1\epsilon_2 c(g,h)).$$

For any subgroup H of $\mathrm{SL}_2(\mathbb{Q}_p)$, we shall denote by $\overline{H}$ the complete inverse image of H in $\widetilde{G}$.

The 2-cocycle c used in the group law above is given by Gelbart; we note that in some references, for example in Manderscheid [11, 12], Baruch–Mao [15, pp. 247–248], and Ishimoto [16], following Kubota, the group law uses the cocycle σ_0. To distinguish between the two, let $\widetilde{G}_c$ denote the realization of the double cover defined by c and let $\widetilde{G}_{\sigma_0}$ denote the realization defined by σ_0. The two are explicitly isomorphic through

$$\iota : \widetilde{G}_c \longrightarrow \widetilde{G}_{\sigma_0}, \qquad \iota((g,\epsilon)) = (g, \epsilon s_p(g)).$$

Indeed, the relation (1) gives

$$\iota((g,\epsilon)(h,\epsilon')) = \iota((g,\epsilon))\iota((h,\epsilon')).$$

We henceforth identify $\widetilde{G}$ with our c-model $\widetilde{G}_c$.

Later in Subsection 4.3 and Section 5, we will use the isomorphism ι when comparing results stated in the two cocycle models.

Let $K = \mathrm{SL}_2(\mathbb{Z}_p)$. By [9, Proposition 2.8] for odd primes p, $\widetilde{G}$ splits over K. Thus $\overline{K}$ is a group isomorphic to the direct product $K \times \mu_2$. We will consider the following subgroups of K:

$$K_0(p^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p) \ : \ c \in p^n\mathbb{Z}_p \right\},$$

$$K_1(p^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p) \ : \ c \in p^n\mathbb{Z}_p, \ a \equiv 1 \pmod{p^n\mathbb{Z}_p} \right\}.$$

For brevity we shall write $K_n := K_0(p^n)$ throughout the paper; the subgroup $K_1(p^n)$ will always be written with its argument. For odd primes p, $\overline{K_0(p^n)}$ is isomorphic to $K_0(p^n) \times \mu_2$ and $\overline{K_1(p^n)}$ is isomorphic to $K_1(p^n) \times \mu_2$.

For the rest of the paper we take p to be an odd prime. In this case, by [9, Corollary 2.13], the center M_p of $\widetilde{G}$ is the direct product $\{\pm I\} \times \mu_2$. Thus a genuine central character is a character of $\{\pm I\} \times \mu_2$ that is non-trivial on the covering-kernel μ_2 factor.

We also fix the Weil-factor conventions used below. For a nontrivial additive character ψ' of $\mathbb{Q}_p$, let $\gamma(\psi')$ denote the Weil index of the one-dimensional quadratic form $x \mapsto x^2$ with respect to ψ'. For $a \in \mathbb{Q}_p^\times$, put $\psi'_a(x) = \psi'(ax)$ and define the relative Weil factor by

$$\gamma(a, \psi') = \frac{\gamma(\psi'_a)}{\gamma(\psi')}.$$

We use the following standard identities; see, for example, [21]:

(1) $\gamma(ac^2, \psi') = \gamma(a, \psi')$,
(2) $\gamma(ab, \psi') = \gamma(a, \psi')\gamma(b, \psi')(a, b)_p$,
(3) $\gamma(a, \psi'_c) = (a, c)_p \gamma(a, \psi')$,
(4) $\gamma(-1, \psi') = \gamma(\psi')^{-2}$,
(5) $\gamma(ac, \psi') = \gamma(c, \psi')$ for any $a \in \mathbb{Z}_p^\times$ and $c \in \mathbb{Q}_p^\times$ such that $\mathrm{ord}_p(c) \equiv c(\psi') \pmod 2$.

Fix a nontrivial additive character ψ of conductor 0 and a nontrivial additive character ψ_- of conductor -1. We put

$$\epsilon_p := \gamma(p, \psi).$$

The factor ϵ_p depends on the fixed additive character ψ; we suppress this dependence in the notation.

We identify $\mathbb{Z}_p/p\mathbb{Z}_p$ with $\mathbb{F}_p$. Whenever representatives are needed in p-adic or classical formulas, we use the integers $0, \ldots, p-1$, and $1, \ldots, p-1$ for $\mathbb{F}_p^\times$.

Let $\chi_p = \left(\frac{\cdot}{p}\right)$, with $\chi_p(0) = 0$, and let $\sqrt{p}$ denote the positive square root of p. Since ψ has conductor 0, the map $t \mapsto \psi(t/p)$ is a well-defined nontrivial additive character of $\mathbb{F}_p$. Its quadratic Gauss sum has the two equivalent expressions

$$\sum_{t\in\mathbb{F}_p^*}\chi_p(t)\psi(t/p)=\sum_{x\in\mathbb{F}_p}\psi(x^2/p)=\epsilon_p\sqrt{p}. \tag{2}$$

The explicit nonarchimedean Weil-index formula gives

$$\gamma(\psi)=1, \qquad \gamma(\psi_p)=p^{-1/2}\sum_{x\in\mathbb{F}_p}\psi(x^2/p);$$

see [21, Theorem A.2(5), Theorem A.3(1)–(2), and Proposition A.11, pp. 366, 369–370]. Since the number of square roots of $t\in\mathbb{F}_p$ is $1+\chi_p(t)$, summing first over the values of x^2, and using the vanishing of a nontrivial additive-character sum, proves the first equality in (2). The displayed Weil-index formulas and the definition of ϵ_p give the second equality. In particular, $\epsilon_p^2=\chi_p(-1)$. We put

$$p^*:=\chi_p(-1)p \qquad \text{and choose} \qquad \sqrt{p^*}:=\epsilon_p\sqrt{p}.$$

For the special choice satisfying $\psi(t/p)=e^{2\pi it/p}$ for $t\in\mathbb{Z}$, one has $\epsilon_p=1$ if $p\equiv 1 \pmod 4$ and $\epsilon_p=i$ if $p\equiv 3 \pmod 4$; see [21, Proposition A.9(5)]. These are the values used in [2]. We do not impose this additional normalization on ψ here.

We set up a few more notations. For $s\in\mathbb{Q}_p$, $t\in\mathbb{Q}_p^\times$ let us define the following elements of $\mathrm{SL}_2(\mathbb{Q}_p)$:

$$x(s)=\begin{pmatrix}1&s\\0&1\end{pmatrix},\ y(s)=\begin{pmatrix}1&0\\s&1\end{pmatrix},\ w(t)=\begin{pmatrix}0&t\\-t^{-1}&0\end{pmatrix},\ h(t)=\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}.$$

We also use the canonical lifts

$$\bar{x}(s):=(x(s),1),\quad \bar{y}(s):=(y(s),1),$$
$$\bar{h}(t):=(h(t),1),\quad \bar{w}(t):=(w(t),1).$$

Let $B_0\subset G$ be the standard upper-triangular Borel subgroup. Let $N=\{(x(s),\epsilon):s\in\mathbb{Q}_p,\ \epsilon=\pm1\}$, $\bar{N}=\{(y(s),\epsilon):s\in\mathbb{Q}_p,\ \epsilon=\pm1\}$ and $T=\{(h(t),\epsilon):t\in\mathbb{Q}_p^\times,\ \epsilon=\pm1\}$ be the subgroups of $\widetilde{G}$. Then the normalizer $N_{\widetilde{G}}(T)$ of T in $\widetilde{G}$ consists of elements $(h(t),\epsilon)$, $(w(t),\epsilon)$ for $t\in\mathbb{Q}_p^\times$. We also put $B=NT=\mathrm{pr}^{-1}(B_0)$. Thus B denotes the full inverse image of the linear Borel B_0. In the notation of Baruch–Mao, our B_0 is their B_S and our B is their $\overline{B}_S$.

Let $n\geq 2$. Following [1], we are interested in the subalgebra of the genuine Hecke algebra of $\widetilde{G}$ modulo $\overline{K_n}$ with quadratic characters whose elements are supported on $\overline{K}$. We define it below.

Let η be a character of K_n such that it is trivial on $K_1(p^n)$. Since $\frac{K_n}{K_1(p^n)}\cong(\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times$, where the quotient map is $\begin{pmatrix}a&b\\c&d\end{pmatrix}\mapsto d \pmod{p^n\mathbb{Z}_p}$, the character η is induced by a character of $(\mathbb{Z}/p^n\mathbb{Z})^\times$. We use the same symbol η for its genuine extension to $\overline{K_n}$, defined by $\eta((A,\epsilon))=\epsilon\eta(A)$ for $A\in K_n$. Recall that a genuine character acts nontrivially on the μ_2 component.

Define $H_n(\eta) := H(\overline{K}//\overline{K_n}, \eta)$ to be the space of functions $f \in C_c^\infty(\widetilde{G})$ supported on $\overline{K}$ and satisfying

$$f(\tilde{k}\tilde{g}\tilde{k}') = \overline{\eta}(\tilde{k})\overline{\eta}(\tilde{k}')f(\tilde{g}) \text{ for } \tilde{g} \in \widetilde{G},\ \tilde{k}, \tilde{k}' \in \overline{K_n}.$$

This is a $\mathbb{C}$-algebra under the usual convolution: for $f_1, f_2 \in H_n(\eta)$,

$$f_1 * f_2(\tilde{h}) = \int_{\widetilde{G}} f_1(\tilde{g})f_2(\tilde{g}^{-1}\tilde{h})d\tilde{g} = \int_{\widetilde{G}} f_1(\tilde{h}\tilde{g})f_2(\tilde{g}^{-1})d\tilde{g},$$

where $d\tilde{g}$ is the Haar measure on $\widetilde{G}$ such that the measure of $\overline{K_n}$ is one.

We take η to be a quadratic character for the rest of the paper. Note that η quadratic character of $(\mathbb{Z}/p^n\mathbb{Z})^\times$ implies that η is either trivial or given by $\left(\frac{\cdot}{p}\right)$. Let $c(\eta)$ denote the conductor exponent of the underlying character of $\mathbb{Z}_p^\times$: it is 0 for the trivial character and 1 for the nontrivial quadratic character modulo p. At level $n = 0$, we take $\eta((k, \epsilon)) = \epsilon$ for $k \in K_0 = K$.

Let (π, V) be a smooth genuine representation of $\widetilde{G}$. For every integer $n \geq c(\eta)$, its $(\overline{K_n}, \eta)$-fixed space is

$$V_\eta^{K_n}(\pi) = \{v \in V : \pi(\tilde{k})v = \eta(\tilde{k})v \text{ for every } \tilde{k} \in \overline{K_n}\}.$$

This space is also called the η-isotypic subspace for $\overline{K_n}$, or the $(\overline{K_n}, \eta)$-isotypic space. Throughout, the superscript K_n refers to the prescribed type under the full inverse image $\overline{K_n}$. We also write $\pi_\eta^{K_n}$ for $V_\eta^{K_n}(\pi)$ when the context is clear. The η-conductor of π is

$$c_\eta(\pi) = \min\{n \geq c(\eta) : V_\eta^{K_n}(\pi) \neq 0\},$$

with $c_\eta(\pi) = \infty$ if this set is empty.

For $F \in H_n(\eta)$ and $v \in V$, define the Hecke action by

$$\pi(F)v := \int_{\widetilde{G}} F(\tilde{g})\pi(\tilde{g})v \, d\tilde{g},$$

using the Haar measure fixed above; compare [14, §3]. By the left equivariance of F, this operator maps all of V into the fixed-type space:

$$\pi(F) : V \longrightarrow V_\eta^{K_n}(\pi).$$

Moreover, $\pi(F_1 * F_2) = \pi(F_1)\pi(F_2)$. Hence $V_\eta^{K_n}(\pi)$ is a left $H_n(\eta)$-module. The same construction applies to the full genuine Hecke algebra $H(\widetilde{G}//\overline{K_n}, \eta)$, where the support need not lie in $\overline{K}$. We write Fv for $\pi(F)v$ when the representation is understood.

If $\mathrm{supp}(F) = \bigsqcup_i \tilde{g}_i\overline{K_n}$, then for $v \in V_\eta^{K_n}(\pi)$ the normalization $\mathrm{vol}(\overline{K_n}) = 1$ gives

$$\pi(F)v = \sum_i F(\tilde{g}_i)\pi(\tilde{g}_i)v.$$

Indeed, on each right coset the factors $\eta(\tilde{k})^{-1}$ from $F(\tilde{g}_i\tilde{k})$ and $\eta(\tilde{k})$ from $\pi(\tilde{g}_i\tilde{k})v$ cancel.

We would like to describe $H_n(\eta)$ using generators and relations. From now on we will write $H = H_n(\eta)$ when the level and character are understood, and use the same notation at any other level at which η is defined.

Proposition 2.1. *A complete set of double coset representatives of* $\overline{\mathrm{SL}_2(\mathbb{Z}_p)}$ *modulo* $\overline{K_n}$ *are given by* $(I,1),\ (w(1),1),\ (y(p^r),1),\ (y(\lambda p^r),1)$ *where* r *runs from* 1 *to* $n-1$ *and* λ *is a nonsquare modulo* p.

The support condition is especially simple for these representatives. Since the chosen cover splits over K, we identify $\overline{K}$ with $K\times\mu_2$ for the following calculation. If $g\in K$, then the double coset $\overline{K_n}(g,1)\overline{K_n}$ supports a nonzero η-biequivariant function if and only if

$$\eta(k)=\eta(g^{-1}kg)\qquad \text{for every } k\in K_n\cap gK_ng^{-1}. \tag{3}$$

Indeed, if k belongs to this intersection, then $kg=g(g^{-1}kg)$, and the two applications of biequivariance must give the same value. Conversely, condition (3) makes the value prescribed at $(g,1)$ independent of the choice of its left–right expression. There is no additional cocycle factor, since all the elements in this calculation lie in the split compact subgroup.

Proposition 2.2. *Every double coset listed in Proposition 2.1 supports a nonzero element of* H.

Proof. The assertion is immediate for $g=I$. Let $g=w(1)$ and let $k=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ belong to $K_n\cap gK_ng^{-1}$. Then both b and c belong to $p^n\mathbb{Z}_p$, and

$$g^{-1}kg=\begin{pmatrix}d&-c\\-b&a\end{pmatrix}.$$

Since $ad-bc=1$, we have $ad\equiv 1 \pmod p$. Hence $\eta(a)=\eta(d^{-1})=\eta(d)$, where the last equality uses that η is quadratic. Thus (3) holds.

Next let $g=y(t)$, where $t=\delta p^r$ with $\delta\in\mathbb{Z}_p^\times$ and $1\le r\le n-1$. For $k=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in K_n\cap gK_ng^{-1}$, direct multiplication gives

$$g^{-1}kg=\begin{pmatrix}a+bt&b\\c+t(d-a)-t^2b&d-tb\end{pmatrix}.$$

Since $t\in p\mathbb{Z}_p$, its lower-right entry is congruent to d modulo p. The character η is either trivial or the quadratic character modulo p, and therefore $\eta(d-tb)=\eta(d)$. This verifies (3) for both square classes $\delta=1$ and $\delta=\lambda$, and proves the proposition. □

We denote by $\mathcal{I}$ and $\mathcal{U}_0$ the elements of H supported on $(I,1)$ and $(w(1),1)$, respectively, normalized by $\mathcal{I}((I,1))=1$ and $\mathcal{U}_0((w(1),1))=1$. For $1\le r\le n-1$ and $a\in\mathbb{Z}_p^\times$, let $\mathcal{V}_{r,a}$ denote the normalized Hecke function supported on $\overline{K_n}(y(ap^r),1)\overline{K_n}$ and satisfying $\mathcal{V}_{r,a}((y(ap^r),1))=1$.

The function $\mathcal{V}_{r,a}$ depends only on the square class of a. Indeed, if $a=bs^{-2}$ with $s\in\mathbb{Z}_p^\times$, then

$$(y(ap^r),1)=(h(s),1)(y(bp^r),1)(h(s)^{-1},1).$$

The two type-character factors associated with $(h(s),1)$ and $(h(s)^{-1},1)$ cancel, and hence $\mathcal{V}_{r,a}=\mathcal{V}_{r,b}$. Consequently, the two possibilities are represented by $\mathcal{V}_{r,1}$ and $\mathcal{V}_{r,\lambda}$.

Note that since η is a character of $\frac{K_n}{K_1(p^n)}$, $\eta(h(u),\epsilon)=\eta(u^{-1})\epsilon$.

We next compute the relations using the notation $K_n = K_0(p^n)$ fixed above.

We will use the following lemma which easily follows from the definition of convolution.

Lemma 2.3. *Let $f_1,\ f_2 \in H$ such that f_1 is supported on $S\tilde{x}S = \bigcup_{i=1}^{m} \tilde{\alpha}_i S = \bigcup_{i=1}^{m} S\tilde{\gamma}_i$ and f_2 is supported on $S\tilde{y}S = \bigcup_{j=1}^{n} \tilde{\beta}_j S = \bigcup_{j=1}^{n} S\tilde{\delta}_j$. Then*

$$f_1 * f_2(\tilde{h}) = \sum_{i=1}^{m} f_1(\tilde{\alpha}_i) f_2(\tilde{\alpha}_i^{-1}\tilde{h}) = \sum_{j=1}^{n} f_1(\tilde{h}\tilde{\delta}_j^{-1}) f_2(\tilde{\delta}_j)$$

where in the first sum the nonzero summands are precisely for those i for which there exist a j such that $\tilde{h} \in \tilde{\alpha}_i\tilde{\beta}_j S$ while in the second sum the nonzero summands are precisely for those j for which there exist a i such that $\tilde{h} \in S\tilde{\gamma}_i\tilde{\delta}_j$.

Proof. We prove the second expression.

$$f_1 * f_2(\tilde{h}) = \int_{\tilde{G}} f_1(\tilde{h}\tilde{g}) f_2(\tilde{g}^{-1}) d\tilde{g}.$$

We have $\tilde{g}^{-1} \in S\tilde{y}S$ if and only if $\tilde{g} \in (S\tilde{y}S)^{-1} = \bigcup_{j=1}^{n} \tilde{\delta}_j^{-1} S$. Hence

$$\begin{aligned} f_1 * f_2(\tilde{h}) &= \sum_{j=1}^{n} \int_{\tilde{\delta}_j^{-1} S} f_1(\tilde{h}\tilde{g}) f_2(\tilde{g}^{-1})\, d\tilde{g} \\ &= \sum_{j=1}^{n} f_1(\tilde{h}\tilde{\delta}_j^{-1}) f_2(\tilde{\delta}_j). \end{aligned}$$

The nonzero summands are precisely those for which $\tilde{h}\tilde{\delta}_j^{-1} \in S\tilde{x}S = \bigcup_{i=1}^{m} S\tilde{\gamma}_i$, equivalently those j for which there is an i with $\tilde{h} \in S\tilde{\gamma}_i\tilde{\delta}_j$.

□

In the following lemma we express the double cosets with our coset representatives into a union of single cosets.

Lemma 2.4.

(1) $K_n w(1) K_n = \bigcup_{s \in \mathbb{Z}_p/p^n\mathbb{Z}_p} x(s) w(1) K_n = \bigcup_{s \in \mathbb{Z}_p/p^n\mathbb{Z}_p} K_n w(1) x(s).$

(2) Let $\lambda \in \mathbb{Z}_p^{\times}$ and $1 \le r \le n-1$. Then

$$\begin{aligned} K_n y(\lambda p^r) K_n &= \bigcup_{u \in \mathbb{Z}_p^*/\sqrt{1+p^{n-r}\mathbb{Z}_p}} h(u) y(\lambda p^r) K_n \\ &= \bigcup_{u \in \mathbb{Z}_p^*/\sqrt{1+p^{n-r}\mathbb{Z}_p}} K_n y(\lambda p^r) h(u). \end{aligned}$$

Proof. (1) follows using triangular decomposition of K_n.
For (2) we use the same idea as [1, Lemmas 3.7, 3.9]. Recall that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_n^{y(\lambda p^r)}$ provided the matrix is in K_n and $a - d + bp^r\lambda \in p^{n-r}\mathbb{Z}_p$. Any element g of the double coset is of the form $x(s)h(u)y(\lambda p^r)K_n = h(u)x(u^{-2}s)y(\lambda p^r)K_n$. If $r \ge n/2$, $x(u^{-2}s) \in K_n^{y(\lambda p^r)}$ and so g is of the

form $h(u)y(\lambda p^r)K_n$. Now $h(u) \in K_n^{y(\lambda p^r)}$ if and only if $u^2 - 1 \in p^{n-r}\mathbb{Z}_p$. Thus g is of the form $h(u)y(\lambda p^r)K_n$ where $u \in \mathbb{Z}_p^*/\sqrt{1+p^{n-r}\mathbb{Z}_p}$. Further, $y(-\lambda p^r)h(u_1u_2^{-1})y(\lambda p^r) \in K_n \iff u_1^{-1}u_2 \in \sqrt{1+p^{n-r}\mathbb{Z}_p}$, thus the union is disjoint.

Now assume that $1 \le r < n/2$. As before, any element g of the double coset can be written as $g = h(u)x(s)y(\lambda p^r)k$ for $s \in \mathbb{Z}_p$, u a unit and $k \in K_n$. Rewrite $g = h(u)h(1+pv)^{-1}h(1+pv)x(s)y(\lambda p^r)k$ where $v \in \mathbb{Z}_p$ such that $h(1+pv)x(s) \in K_n^{y(\lambda p^r)}$; this is possible by taking $v = \frac{\sqrt{(1+s\lambda p^r)^{-1}}-1}{p}$ (we take the square root in $1+p^r\mathbb{Z}_p$ so that $v \in \mathbb{Z}_p$). Thus g is of the form $h(u')y(\lambda p^r)K_n$ and we can follow the argument as above.

A similar argument gives the right coset decomposition.

$\square$

Remark. We record a correction to the proof of [1, Lemma 3.9]. This concerns the integral-weight setting of that paper, where $K_0 = K_0(p^n) \subset \mathrm{GL}_2(\mathbb{Z}_p)$ and $d(s) = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}$. Put $t = p^r$ and write the diagonal parameter appearing at the beginning of that proof as s_0. The assertion there that $d(1+tu)x(u)$ belongs to $K_0^{y(t)} = y(t)K_0y(-t) \cap K_0$ is not valid in general. Instead, set

$$s = (1+ut)^{-1} \in \mathbb{Z}_p^\times.$$

Then direct multiplication gives

$$y(-t)d(s)x(u)y(t) = \begin{pmatrix} s(1+ut) & su \\ t\{1-s(1+ut)\} & 1-sut \end{pmatrix} = \begin{pmatrix} 1 & su \\ 0 & 1-sut \end{pmatrix} \in K_0.$$

Consequently $d(s)x(u)y(t) = y(t)k_1$ for some $k_1 \in K_0$, and hence

$$\begin{aligned} g &= d(s_0)d(s)^{-1}d(s)x(u)y(t)k_0 \\ &= d(s_0s^{-1})y(t)k_1k_0. \end{aligned}$$

This proves the required coset decomposition; the disjointness argument in [1, Lemma 3.9] is unchanged.

We note the following simple observation.

Lemma 2.5. *We have* $\sqrt{1+p^r\mathbb{Z}_p} = (1+p^r\mathbb{Z}_p)\bigcup(-1+p^r\mathbb{Z}_p)$. *In particular the coset representatives of* $\mathbb{Z}_p^*/\sqrt{1+p^r\mathbb{Z}_p}$ *are given by* $x_0+x_1p+\cdots+x_{r-1}p^{r-1}$ *where* $x_0 \in \{1,2,\ldots,(p-1)/2\}$ *and* $x_1,\ldots,x_{r-1} \in \{0,1,\ldots,p-1\}$ *and so* $\#\left(\mathbb{Z}_p^*/\sqrt{1+p^r\mathbb{Z}_p}\right) = \frac{p-1}{2}p^{r-1}$.

Proof. Let $a \in \sqrt{1+p^r\mathbb{Z}_p}$. Then $a^2 \in 1+p^r\mathbb{Z}_p$. In particular $a = \pm 1$ modulo p. Let $a = 1+x_1p+x_2p^2+\cdots+x_rp^r$ with $x_r \in \mathbb{Z}_p$ and $x_1,\ldots,x_{r-1} \in \{0,1,\ldots,p-1\}$. Then $a^2 = 1+2x_1p$ modulo $p^2\mathbb{Z}_p$ which implies that $x_1 = 0$. Continuing this argument, we get that $a \in 1+p^r\mathbb{Z}_p$. If $a \equiv -1 \pmod{p\mathbb{Z}_p}$, the same argument will imply that $a \in (-1+p^r\mathbb{Z}_p)$. $\square$

3. Relations

The algebra $H = H_n(\eta) = H(\overline{K}//\overline{K_n},\eta)$, where η is a quadratic character modulo p (either the trivial character or $\left(\frac{\cdot}{p}\right)$), has the following $2n$ basis elements:

$$\mathcal{I},\quad \mathcal{U}_0,\quad \mathcal{V}_{1,1},\dots,\mathcal{V}_{n-1,1},\quad \mathcal{V}_{1,\lambda},\dots,\mathcal{V}_{n-1,\lambda}.$$

Here λ is a nonsquare modulo p, and $\mathcal{I}$ is the identity element.

Throughout this section all the matrices occurring in the convolution calculations belong to K. We therefore use the fixed splitting over K and identify $\overline{K}$ with $K \times \mu_2$. In particular, products and inverses of the canonical lifts appearing below introduce no additional cocycle factor.

We will use the following well-known result.

Lemma 3.1. *Let p be an odd prime, and let a, b be integers such that $p \nmid ab$. Then the number of solutions (x, y) to $ax^2 + by^2 \equiv 1 \pmod p$ is $p - \left(\frac{-ab}{p}\right)$.*

Proposition 3.2. *Suppose $p \equiv 1 \pmod 4$. Then*

$$\mathcal{V}_{r,1} * \mathcal{V}_{r,1} = \frac{p-1}{2}p^{n-r-1}\mathcal{Y}_{r+1} + \frac{p-5}{4}p^{n-r-1}\mathcal{V}_{r,1} + \frac{p-1}{4}p^{n-r-1}\mathcal{V}_{r,\lambda}$$

$$\mathcal{V}_{r,\lambda} * \mathcal{V}_{r,\lambda} = \frac{p-1}{2}p^{n-r-1}\mathcal{Y}_{r+1} + \frac{p-1}{4}p^{n-r-1}\mathcal{V}_{r,1} + \frac{p-5}{4}p^{n-r-1}\mathcal{V}_{r,\lambda}$$

$$\mathcal{V}_{r,1} * \mathcal{V}_{r,\lambda} = \mathcal{V}_{r,\lambda} * \mathcal{V}_{r,1} = \frac{p-1}{4}p^{n-r-1}(\mathcal{V}_{r,1} + \mathcal{V}_{r,\lambda}).$$

Suppose $p \equiv 3 \pmod 4$. Then

$$\mathcal{V}_{r,1} * \mathcal{V}_{r,1} = \frac{p-3}{4}p^{n-r-1}\mathcal{V}_{r,1} + \frac{p+1}{4}p^{n-r-1}\mathcal{V}_{r,\lambda}$$

$$\mathcal{V}_{r,\lambda} * \mathcal{V}_{r,\lambda} = \frac{p+1}{4}p^{n-r-1}\mathcal{V}_{r,1} + \frac{p-3}{4}p^{n-r-1}\mathcal{V}_{r,\lambda}$$

$$\mathcal{V}_{r,1} * \mathcal{V}_{r,\lambda} = \mathcal{V}_{r,\lambda} * \mathcal{V}_{r,1} = \frac{p-1}{2}p^{n-r-1}\mathcal{Y}_{r+1} + \frac{p-3}{4}p^{n-r-1}(\mathcal{V}_{r,1} + \mathcal{V}_{r,\lambda}).$$

Where in both cases $\mathcal{Y}_r = \mathcal{I} + \sum_{j=r}^{n-1} \mathcal{V}_j$ with $\mathcal{V}_j = \mathcal{V}_{j,1} + \mathcal{V}_{j,\lambda}$ and $\mathcal{Y}_n = \mathcal{I}$.

Proof. Let $\delta, \mu \in \{1, \lambda\}$. It follows from Lemmas 2.3 and 2.4 that $\mathcal{V}_{r,\delta} * \mathcal{V}_{r,\mu}$ is supported on those $(g, 1) \in \bar{K}$ for which there exist $u,\ v \in \mathbb{Z}_p^*/\sqrt{1 + p^{n-r}\mathbb{Z}_p}$ such that

$$(h(u)y(\delta p^r)h(v)y(\mu p^r))^{-1}g = \begin{pmatrix} (uv)^{-1} & 0 \\ -p^r u^{-1}(\delta v + \mu v^{-1}) & uv \end{pmatrix} g \in K_n.$$

We check this condition for $g = I,\ w(1),\ y(p^j), y(\lambda p^j)$ with $j < r$ and $j \geq r$.

For $g = w(1)$, the condition clearly does not hold for any value of u, v, so no support.

For $g = I$, the condition becomes $v^2 + \mu\delta^{-1} \equiv 0 \pmod p$. So if $\mu = \delta$, such v exists if and only if $p \equiv 1 \pmod 4$, while if $\mu \neq \delta$, such v exists if and only if $p \equiv 3 \pmod 4$.

For $g = y(p^j\xi)$ with $\xi \in \{1, \lambda\}$ and $j < r$ the condition becomes $uvp^j\xi - p^r u^{-1}(\delta v + \mu v^{-1}) \equiv 0 \pmod{p^n} \iff uv\xi - p^{r-j}u^{-1}(\delta v + \mu v^{-1}) \equiv 0 \pmod{p^{n-j}}$ which is obviously never satisfied.

For $g = y(p^r)$, the condition is equivalent to $\mu^{-1}(uv)^2 - \delta\mu^{-1}v^2 \equiv 1 \pmod p$ by Hensel's Lemma. By Lemma 3.1, the number of solutions to $\mu^{-1}x^2 - \delta\mu^{-1}y^2 \equiv 1 \pmod p$ equals $p - \left(\frac{\delta}{p}\right)$. For support, we are interested in existence of such (x, y) with x, y both units. If $\delta = \mu = 1$ then the number of such (x, y) is $p - 5$ if $p \equiv 1 \pmod 4$, while it is $p - 3$ if $p \equiv 3 \pmod 4$. If $\delta = \mu = \lambda$ then the number of such (x, y) is $p - 1$ if $p \equiv 1 \pmod 4$, while it is $p + 1$ if $p \equiv 3 \pmod 4$. If $\mu \neq \delta$, then the number of such (x, y) is $p - 1$ if $p \equiv 1 \pmod 4$, while it is $p - 3$ if $p \equiv 3 \pmod 4$.

For $g = y(\lambda p^r)$, the condition is equivalent to $\lambda\mu^{-1}(uv)^2 - \delta\mu^{-1}v^2 \equiv 1 \pmod p$. By Lemma 3.1, the number of solutions to $\lambda\mu^{-1}x^2 - \delta\mu^{-1}y^2 \equiv 1 \pmod p$ equals $p - \left(\frac{\lambda\delta}{p}\right)$. As before we are only interested in solutions (x, y) with x, y both units. If $\delta = \mu = 1$ then the number of such (x, y) is $p - 1$ if $p \equiv 1 \pmod 4$, while it is $p + 1$ if $p \equiv 3 \pmod 4$. If $\delta = \mu = \lambda$ then the number of such (x, y) is $p - 5$ if $p \equiv 1 \pmod 4$, while it is $p - 3$ if $p \equiv 3 \pmod 4$. If $\mu \neq \delta$, then the number of such (x, y) is $p - 1$ if $p \equiv 1 \pmod 4$, while it is $p - 3$ if $p \equiv 3 \pmod 4$.

For $g = y(p^j\xi)$ with $\xi \in \{1, \lambda\}$ and $j > r$ the condition becomes $(u^2p^{j-r}\xi - \delta)v^2 - \mu \equiv 0 \pmod{p^{n-r}}$ which has a solution if and only if $\delta v^2 + \mu \equiv 0 \pmod p$. This implies as before that either $\mu = \delta$ and $p \equiv 1 \pmod 4$ or $\mu \neq \delta$ and $p \equiv 3 \pmod 4$.

Thus, for $p \geq 7$, $\mathcal{V}_{r,\delta} * \mathcal{V}_{r,\delta}$ is supported on $(I, 1)$, $(y(p^j), 1)$, $(y(\lambda p^j), 1)$ with $j \geq r$ if $p \equiv 1 \pmod 4$, while supported on $(y(p^r), 1)$, $(y(\lambda p^r), 1)$ if $p \equiv 3 \pmod 4$. On the other hand, for $p \geq 7$, $\mathcal{V}_{r,\delta} * \mathcal{V}_{r,\mu}$, for $\delta \neq \mu$, is supported on $(y(p^r), 1)$ and $(y(\lambda p^r), 1)$ if $p \equiv 1 \pmod 4$, while it is supported on $(I, 1)$ and $(y(p^j), 1)$, $(y(\lambda p^j), 1)$ with $j \geq r$ if $p \equiv 3 \pmod 4$. For $p = 3$, $\mathcal{V}_{r,1} * \mathcal{V}_{r,1}$ is only supported on $(y(\lambda p^r), 1)$, $\mathcal{V}_{r,\lambda} * \mathcal{V}_{r,\lambda}$ is only supported on $(y(p^r), 1)$ while $\mathcal{V}_{r,\delta} * \mathcal{V}_{r,\mu}$, for $\delta \neq \mu$ is supported on $(I, 1)$ and $(y(p^j), 1)$, $(y(\lambda p^j), 1)$ with $j \geq r + 1$. For $p = 5$, $\mathcal{V}_{r,1} * \mathcal{V}_{r,1}$ is supported on $(I, 1)$, $(y(\lambda p^r), 1)$ and on $(y(p^j), 1)$, $(y(\lambda p^j), 1)$ with $j \geq r + 1$; $\mathcal{V}_{r,\lambda} * \mathcal{V}_{r,\lambda}$ is supported on $(I, 1)$, $(y(p^r), 1)$ and on $(y(p^j), 1)$, $(y(\lambda p^j), 1)$ with $j \geq r + 1$; while $\mathcal{V}_{r,\delta} * \mathcal{V}_{r,\mu}$, for $\delta \neq \mu$, is supported on $(y(p^r), 1)$ and $(y(\lambda p^r), 1)$. Thus, the support agrees with the relations in the statement.

Now we compute the values of $\mathcal{V}_{r,\delta} * \mathcal{V}_{r,\mu}$ on the coset representatives where they are supported.

We have

$$\begin{aligned}\mathcal{V}_{r,\delta} * \mathcal{V}_{r,\mu}((g,1)) &= \sum_{u \in \mathbb{Z}_p^*/\sqrt{1+p^{n-r}\mathbb{Z}_p}} \mathcal{V}_{r,\delta}((h(u),1)(y(\delta p^r),1)) \\ &\qquad\qquad \cdot \mathcal{V}_{r,\mu}((y(-\delta p^r),1)(h(u^{-1}),1)(g,1)) \\ &= \sum_{u \in \mathbb{Z}_p^*/\sqrt{1+p^{n-r}\mathbb{Z}_p}} \overline{\eta}(u^{-1}) \\ &\qquad\qquad \cdot \mathcal{V}_{r,\mu}((y(-\delta p^r),1)(h(u^{-1}),1)(g,1)).\end{aligned}$$

By the splitting over K,

$$(y(-\delta p^r),1)(h(u^{-1}),1) = \left(\begin{pmatrix} u^{-1} & 0 \\ -\delta p^r u^{-1} & u \end{pmatrix}, 1\right),$$

so

$$\mathcal{V}_{r,\delta} * \mathcal{V}_{r,\mu}((g,1)) = \sum_{u \in \mathbb{Z}_p^*/\sqrt{1+p^{n-r}\mathbb{Z}_p}} \overline{\eta}(u^{-1})\mathcal{V}_{r,\mu}((\begin{pmatrix} u^{-1} & 0 \\ -\delta p^r u^{-1} & u \end{pmatrix}, 1)(g,1)).$$

Let $g = I$ and write $k_1 = \begin{pmatrix} u^{-1} & 0 \\ -\delta p^r u^{-1} & u \end{pmatrix}$. We seek $A, k_2 \in K_n$ such that

$$(k_1, 1) = (A^{-1}, 1)(y(\mu p^r), 1)(k_2, 1).$$

We find $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_n$ such that $y(-\mu p^r) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ -\delta p^r u^{-1} & u \end{pmatrix} \in K_n$, i.e.,

$$\begin{pmatrix} au^{-1} - b\delta p^r u^{-1} & bu \\ -p^r u^{-1}(\mu a + d\delta) + cu^{-1} + p^{2r}\mu\delta b u^{-1} & du \end{pmatrix} \in K_n.$$

This boils down to finding a such that $a^2 + \delta\mu^{-1} \equiv 0 \pmod{p}$.

Let $\delta = \mu$. If $p \equiv 3 \pmod 4$, no such a exist. If $p \equiv 1 \pmod 4$, take $a = \sqrt{-1}$, $d = 1/a$, $b = c = 0$. For this choice of A, $k_2 = h(au^{-1})$. Thus,

$$\begin{aligned}
\mathcal{V}_{r,\delta} * \mathcal{V}_{r,\mu}((I,1)) &= \sum_{u \in \mathbb{Z}_p^*/\sqrt{1+p^{n-r}\mathbb{Z}_p}} \overline{\eta}(u^{-1})\mathcal{V}_{r,\mu}((k_1,1)) \\
&= \sum_{u \in \mathbb{Z}_p^*/\sqrt{1+p^{n-r}\mathbb{Z}_p}} \overline{\eta}(u^{-1}) \\
&\qquad \cdot \mathcal{V}_{r,\mu}((h(1/\sqrt{-1}),1)(y(\mu p^r),1)(h(u^{-1}\sqrt{-1}),1)) \\
&= \sum_{u \in \mathbb{Z}_p^*/\sqrt{1+p^{n-r}\mathbb{Z}_p}} \overline{\eta}(u^{-1})\overline{\eta}(u) \\
&= \#\{u \in \mathbb{Z}_p^*/\sqrt{1+p^{n-r}\mathbb{Z}_p}\} = \frac{p-1}{2}p^{n-r-1}.
\end{aligned}$$

Let $\delta \neq \mu$. So $\delta\mu^{-1}$ is a non-square modulo p. If $p \equiv 1 \pmod 4$, no such a exist. If $p \equiv 3 \pmod 4$, take $a = \sqrt{-\delta\mu^{-1}}$, $d = 1/a$, $b = c = 0$; in this case, as before, $\mathcal{V}_{r,\delta} * \mathcal{V}_{r,\mu}((I,1)) = \frac{p-1}{2}p^{n-r-1}$.

Let $g = y(\beta p^r)$ where $\beta \in \{1, \lambda\}$. Then,

$$\begin{aligned}
(y(-\delta p^r),1)(h(u^{-1}),1)(y(\beta p^r),1) &= \left(\begin{pmatrix} u^{-1} & 0 \\ -\delta p^r u^{-1} & u \end{pmatrix}, 1\right)(y(\beta p^r),1) \\
&= \left(\begin{pmatrix} u^{-1} & 0 \\ p^r u^{-1}\beta(u^2 - \frac{\delta}{\beta}) & u \end{pmatrix}, 1\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{V}_{r,\delta} * \mathcal{V}_{r,\mu}((y(\beta p^r),1)) &= \sum_{u \in \mathbb{Z}_p^*/\sqrt{1+p^{n-r}\mathbb{Z}_p}} \overline{\eta}(u^{-1}) \\
&\qquad \cdot \mathcal{V}_{r,\mu}\left(\left(\begin{pmatrix} u^{-1} & 0 \\ p^r u^{-1}\beta(u^2 - \frac{\delta}{\beta}) & u \end{pmatrix}, 1\right)\right).
\end{aligned}$$

For $u \in \mathbb{Z}_p^*/\sqrt{1+p^{n-r}\mathbb{Z}_p}$, put $v = u^{-1}\beta(u^2 - \frac{\delta}{\beta})$. We want $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_n$ such that

$$y(-\mu p^r)A\begin{pmatrix} u^{-1} & 0 \\ -\delta p^r u^{-1} + \beta p^r u & u \end{pmatrix} = \begin{pmatrix} au^{-1} + bp^r v & bu \\ * & du - \mu p^r bu \end{pmatrix} \in K_n,$$

where

$$* = -\mu p^r(au^{-1} + bp^r v) + cu^{-1} + p^r dv.$$

This simplifies to finding a, u units in $\mathbb{Z}_p$ such that

$$-\frac{\mu}{\delta}a^2 + \frac{\beta}{\delta}u^2 = 1 \pmod{p^{n-r}}.$$

Suppose for some u, we have a solution a mod p as above, then we can lift this to $a \in \mathbb{Z}_p$ and take $A = h(a)$ so that the matrix above becomes $h(au^{-1})$. For such u we verify that

$$(\begin{pmatrix} u^{-1} & 0 \\ p^r u^{-1}\beta(u^2 - \frac{\delta}{\beta}) & u \end{pmatrix}, 1) = (h(a^{-1}), 1)(y(\mu p^r), 1)(h(au^{-1}), 1),$$

and so the contribution from this u in the sum for $\mathcal{V}_{r,\delta} * \mathcal{V}_{r,\mu}((y(\beta p^r), 1))$ would be $\overline{\eta}(u^{-1})\overline{\eta}(a)\overline{\eta}(a^{-1}u)$ which equals 1.

So, we need to just count the number of solutions (a, u) with a, u units to the above congruence modulo p^{n-r}. We know however the number of solutions (mod p) by Lemma 3.1. Since for u there are four possibilities $(\pm a, \pm u)$ and $u \in \mathbb{Z}_p^*/\sqrt{1+p\mathbb{Z}_p}$, we will divide by 4 to get the contribution.

The counting is similar to the counting that we did while checking the support, so we will illustrate it only for one case. By Lemma 3.1, the total number of (a, u) mod p (not necessarily units) is $p - \left(\frac{\mu\beta}{p}\right)$. Let $\mu = \beta$ and $p \equiv 1 \pmod 4$. In this case, solutions of the type $(\pm *, 0)$ exists if and only if $\mu = \delta$. Also, solutions of the type $(0, \pm *)$ exists if and only if $\beta = \delta$. So, if $\beta = \mu = \delta$, then the total number of solutions (a, u) with both a, u units equals $p - 1 - 4 = p - 5$. For each $u \in \mathbb{Z}_p^*/\sqrt{1+p^{n-r}\mathbb{Z}_p}$ however that satisfies $-\frac{\mu}{\delta}a^2 + \frac{\beta}{\delta}u^2 = 1 \pmod p$ we can find a solution $a \in \mathbb{Z}_p$ and thus there are $\#\{u \in \mathbb{Z}_p^*/\sqrt{1+p^{n-r}\mathbb{Z}_p} | \exists a \in \mathbb{Z}_p : -\frac{\mu}{\delta}a^2 + \frac{\beta}{\delta}u^2 = 1 \pmod p\} = \frac{p-5}{4}p^{n-r-1}$ solutions in total. Hence, $\mathcal{V}_{r,1} * \mathcal{V}_{r,1}((y(p^r), 1)) = \frac{p-5}{4}p^{n-r-1}$ if $p \equiv 1 \pmod 4$. The other cases follow similarly.

For $g = (y(p^j\beta), 1)$ with $j > r$ and $\beta \in \{1, \lambda\}$ we want that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_n$ such that

$$y(-\mu p^r)A\begin{pmatrix} u^{-1} & 0 \\ -\delta p^r u^{-1} + \beta p^j u & u \end{pmatrix} \in K_n.$$

This is equivalent to the condition $-\mu a + (d - \mu p^r b)(\beta p^{j-r}u^2 - \delta) \equiv 0 \pmod{p^{n-r}}$ which is equivalent to lifting a solution of $a^2 + \delta\mu^{-1} \equiv 0 \pmod p$. The rest of the proof now follows as in the case of $g = (I, 1)$. □

Proposition 3.3.

$$\mathcal{V}_{r,\mu} * \mathcal{V}_{j,\delta} = \mathcal{V}_{j,\delta} * \mathcal{V}_{r,\mu} = \frac{p-1}{2}p^{n-j-1}\mathcal{V}_{r,\mu}, \ \textit{for } j > r.$$

Proof. Assume $j > r$, then by Lemmas 2.3 and 2.4, $\mathcal{V}_{r,\mu} * \mathcal{V}_{j,\delta}$ is supported on the $(g,1)$ for which there exist $u \in \mathbb{Z}_p^*/\sqrt{1+p^{n-r}\mathbb{Z}_p}$ and $v \in \mathbb{Z}_p^*/\sqrt{1+p^{n-j}\mathbb{Z}_p}$ such that

$$\begin{pmatrix} (uv)^{-1} & 0 \\ -p^r u^{-1} v\mu - p^j (uv)^{-1}\delta & uv \end{pmatrix} g \in K_n.$$

It is obvious then that for $g = w(1)$ there is no support.
For $g = y(p^i\beta)$ we want equivalently $uvp^i\beta - p^r u^{-1} v\mu - p^j(uv)^{-1}\delta \in p^n\mathbb{Z}_p$. For $i \neq r$ it is easy to check that this is impossible. For $i = r$ the condition becomes $uv\beta - u^{-1}v\mu - p^{j-r}(uv)^{-1}\delta \in p^{n-r}\mathbb{Z}_p$. This is equivalent to finding a solution to $u^2 - \beta^{-1}\mu \equiv 0 \pmod p$ which exists if and only if $\beta = \mu$. We thus calculate the value for $g = y(p^r\mu)$.
As before

$$\begin{aligned} \mathcal{V}_{r,\mu} * \mathcal{V}_{j,\delta}((y(p^r\mu),1)) = &\sum_{u \in \mathbb{Z}_p^*/\sqrt{1+p^{n-r}\mathbb{Z}_p}} \overline{\eta}(u^{-1}) \\ &\cdot \mathcal{V}_{j,\delta}((y(-p^r\mu)h(u^{-1})y(p^r\mu),1)). \end{aligned}$$

We want $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_n$ such that

$$y(-\delta p^j)A \begin{pmatrix} u^{-1} & 0 \\ \mu p^r(u-u^{-1}) & u \end{pmatrix} \in K_n.$$

A direct multiplication shows that this is equivalent to

$$\mu(d - p^j b\delta)(u-u^{-1}) - \delta p^{j-r} a u^{-1} \in p^{n-r}\mathbb{Z}_p.$$

Thus necessarily $u - u^{-1} = p^{j-r}\zeta$ with $\zeta \in \mathbb{Z}_p^\times$. Taking $b = 0$ and using $ad = 1$, the remaining congruence is

$$\mu d^2 \zeta u \equiv \delta \pmod{p^{n-j}}.$$

Hence, writing $\zeta = (u-u^{-1})/p^{j-r}$, the contribution is

$$\begin{aligned} \mathcal{V}_{r,\mu} * \mathcal{V}_{j,\delta}((y(p^r\mu),1)) &= \#\left\{ u \in \mathbb{Z}_p^*/\sqrt{1+p^{n-r}\mathbb{Z}_p} \;\middle|\; \zeta \in \mathbb{Z}_p^\times, u \in \frac{\mu\zeta}{\delta}(\mathbb{Z}_p^\times)^2 \right\} \\ &= \frac{p-1}{2} p^{n-j-1}. \end{aligned}$$

From the second part of Lemma 2.3,

$$\begin{aligned} \mathcal{V}_{j,\delta} * \mathcal{V}_{r,\mu}((y(p^r\mu),1)) = &\sum_{u \in \mathbb{Z}_p^*/\sqrt{1+p^{n-r}\mathbb{Z}_p}} \overline{\eta}(u^{-1}) \\ &\cdot \mathcal{V}_{j,\delta}((y(p^r\mu)h(u)y(-p^r\mu),1)) \\ = &\mathcal{V}_{r,\mu} * \mathcal{V}_{j,\delta}((y(p^r\mu),1)). \end{aligned}$$

□

Proposition 3.4. *If $\eta = 1$ then with notation as before*

$$\mathcal{U}_0 * \mathcal{U}_0 = p^n \mathcal{Y}_1 + p^{n-1}(p-1)\mathcal{U}_0.$$

If $\eta \neq 1$ then

$$\mathcal{U}_0 * \mathcal{U}_0 = \eta(-1) p^n \mathcal{Y}_1.$$

Proof. By Lemmas 2.3 and 2.4,

$$\begin{aligned}\mathcal{U}_0 * \mathcal{U}_0((g,1)) &= \sum_{s\in\mathbb{Z}_p/p^n\mathbb{Z}_p} \mathcal{U}_0((x(s),1)(w(1),1)) \\ &\qquad\qquad \cdot \mathcal{U}_0((w(-1),1)(x(-s),1)(g,1)) \\ &= \sum_{s\in\mathbb{Z}_p/p^n\mathbb{Z}_p} \mathcal{U}_0((\begin{pmatrix}0 & -1\\ 1 & -s\end{pmatrix},1)(g,1)).\end{aligned}$$

where the support of $\mathcal{U}_0 * \mathcal{U}_0$ is on those $(g,1)$ for which there exist $s,t\in\mathbb{Z}_p$ such that $y(t)x(-s)g\in K_n$. It is easy to check that for $g = I, w(1), y(\beta p^r)$ some such s,t exist.

Let $g = I$. Note that $(\begin{pmatrix}0 & -1\\ 1 & -s\end{pmatrix},1) = (w(1),1)(\begin{pmatrix}-1 & s\\ 0 & -1\end{pmatrix},1)$. Hence, $\mathcal{U}_0 * \mathcal{U}_0((I,1)) = p^n\eta(-1)$.

Let $g = w(1)$. Then $(\begin{pmatrix}0 & -1\\ 1 & -s\end{pmatrix},1)(g,1) = (y(s),1)$. To compute $\mathcal{U}_0 * \mathcal{U}_0$ at $(w(1),1)$, for $s\in\mathbb{Z}_p/p^n\mathbb{Z}_p$, we want $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in K_n$ such that $w(-1)\begin{pmatrix}a & b\\ c & d\end{pmatrix}y(s) = \begin{pmatrix}-c-ds & -d\\ a+bs & b\end{pmatrix}\in K_n$. Clearly s must be in $\mathbb{Z}_p^*$ and for such s, choosing $a=d=1, c=0, b=-1/s$, the above matrix becomes $\begin{pmatrix}-s & -1\\ 0 & -1/s\end{pmatrix}$. Then

$$\begin{aligned}\mathcal{U}_0 * \mathcal{U}_0((w(1),1)) &= \sum_{s\in\mathbb{Z}_p/p^n\mathbb{Z}_p} \mathcal{U}_0((y(s),1)) \\ &= \sum_{\substack{s\in\mathbb{Z}_p/p^n\mathbb{Z}_p\\ (s,p)=1}} \mathcal{U}_0\left((x(1/s),1)(w(1),1)\left(\begin{pmatrix}-s & -1\\ 0 & -1/s\end{pmatrix},1\right)\right) \\ &= \sum_{\substack{s\in\mathbb{Z}_p/p^n\mathbb{Z}_p\\ (s,p)=1}} \overline{\eta}(-1/s) = \begin{cases} p^n - p^{n-1}, & \eta = 1,\\ 0, & \eta = \left(\frac{\cdot}{p}\right).\end{cases}\end{aligned}$$

Let $g = y(\beta p^r)$ where $\beta\in\{1,\lambda\}$. Then

$$(\begin{pmatrix}0 & -1\\ 1 & -s\end{pmatrix},1)(g,1) = (\begin{pmatrix}-\beta p^r & -1\\ 1-\beta p^r s & -s\end{pmatrix},1).$$

Doing computations as above, we find that

$$\begin{aligned}\mathcal{U}_0 * \mathcal{U}_0((y(\beta p^r),1)) &= \sum_{s\in\mathbb{Z}_p/p^n\mathbb{Z}_p} \mathcal{U}_0\left(\left(\left(\begin{smallmatrix}-\beta p^r & -1\\ 1-\beta p^r s & -s\end{smallmatrix}\right),1\right)\right) \\ &= \sum_{s\in\mathbb{Z}_p/p^n\mathbb{Z}_p} \mathcal{U}_0\left(\begin{array}{c}\left(x\left(\dfrac{-\beta p^r}{1-\beta p^r s}\right),1\right)\\ \cdot (w(1),1)\\ \cdot \left(\left(\begin{smallmatrix}-(1-\beta p^r s) & s\\ 0 & -1/(1-\beta p^r s)\end{smallmatrix}\right),1\right)\end{array}\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{s \in \mathbb{Z}_p/p^n\mathbb{Z}_p} \overline{\eta}\left(\frac{-1}{1-\beta p^r s}\right)\\
&= p^n \eta(-1).
\end{aligned}
$$

□

Proposition 3.5. *Let $\delta \in \{1, \lambda\}$. Then*

$$\mathcal{U}_0 * \mathcal{V}_{r,\delta} = \frac{p-1}{2} p^{n-r-1} \mathcal{U}_0 = \mathcal{V}_{r,\delta} * \mathcal{U}_0.$$

Proof. We use Lemmas 2.3 and 2.4. For $\mathcal{U}_0 * \mathcal{V}_{r,\delta}$, we use the first expression in Lemma 2.3 while for $\mathcal{V}_{r,\delta} * \mathcal{U}_0$, we use the second sum. For $s \in \mathbb{Z}_p, t \in \mathbb{Z}_p^*$, note that $(x(s)w(1)h(t)y(\delta p^r))^{-1} = \begin{pmatrix} 0 & -t^{-1} \\ t & \delta p^r t^{-1} - ts \end{pmatrix}$ while $(y(\delta p^r)h(t)w(1)x(s))^{-1} = \begin{pmatrix} \delta p^r t - t^{-1}s & -t \\ t^{-1} & 0 \end{pmatrix}$. It follows that $\mathcal{U}_0 * \mathcal{V}_{r,\delta}$ and $\mathcal{V}_{r,\delta} * \mathcal{U}_0$ are only supported on $(w(1), 1)$.

We first compute $\mathcal{U}_0 * \mathcal{V}_{r,\delta}((w(1),1))$. For $s \in \mathbb{Z}_p/p^n\mathbb{Z}_p$, we want $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_n$ such that $y(-\delta p^r)\begin{pmatrix} a & b \\ c & d \end{pmatrix} y(s) \in K_n$. So, we must have $(d - \delta p^r b)s - \delta p^r a \equiv 0 \pmod{p^n}$. So, s must be of the form $p^r u$ where u is a unit such that u/δ is a square modulo p. Note that there are $\frac{p-1}{2}p^{n-r-1}$ such s in $\mathbb{Z}_p/p^n\mathbb{Z}_p$.

For each such $s = p^r u$, take $a = \sqrt{u/\delta}$, $d = 1/a$, $b = c = 0$. Then,

$$
\begin{aligned}
\mathcal{U}_0 * \mathcal{V}_{r,\delta}((w(1),1)) &= \sum_{s \in \mathbb{Z}_p/p^n\mathbb{Z}_p} \mathcal{U}_0((x(s),1)(w(1),1)) \\
&\qquad \cdot \mathcal{V}_{r,\delta}((w(-1),1)(x(-s),1)(w(1),1)) \\
&= \sum_{s \in \mathbb{Z}_p/p^n\mathbb{Z}_p} \mathcal{V}_{r,\delta}((y(s),1)) \\
&= \sum_{\substack{s \in \mathbb{Z}_p/p^n\mathbb{Z}_p \\ s = p^r u:\ \left(\frac{u/\delta}{p}\right) = 1}} \mathcal{V}_{r,\delta}\big((h(\sqrt{\delta/u}),1)(y(\delta p^r),1) \\
&\qquad \cdot (h(\sqrt{u/\delta}),1)\big) \\
&= \frac{p-1}{2} p^{n-r-1}.
\end{aligned}
$$

We also have however

$$
\begin{aligned}
\mathcal{V}_{r,\delta} * \mathcal{U}_0((w(1),1)) &= \sum_{s \in \mathbb{Z}_p/p^n\mathbb{Z}_p} \mathcal{V}_{r,\delta}((w(1),1)(x(-s),1)(w(-1),1)) \\
&\qquad \cdot \mathcal{U}_0((w(1),1)(x(s),1)) \\
&= \sum_{s \in \mathbb{Z}_p/p^n\mathbb{Z}_p} \mathcal{V}_{r,\delta}((y(-s),1)) = \frac{p-1}{2} p^{n-r-1}.
\end{aligned}
$$

where the last equality follows as before.

□

Theorem 1. *The algebra $H(\overline{K}//\overline{K_n},\eta)$ is a $2n$-dimensional commutative algebra with generators $\{\mathcal{I},\ \mathcal{U}_0,\ \mathcal{V}_{i,1},\ \mathcal{V}_{i,\lambda}\}$ with $1 \le i \le n-1$ where $\left(\frac{\lambda}{p}\right) = -1$ and relations given by Propositions 3.2, 3.3, 3.4, 3.5.*

Define $\mathcal{W}_r := \mathcal{V}_{r,1} - \mathcal{V}_{r,\lambda}$. It follows from Propositions 3.2, 3.3, 3.4 and 3.5 that

Corollary 3.6. *(1) $\mathcal{V}_r^2 = (p-2)p^{n-r-1}\mathcal{V}_r + (p-1)p^{n-r-1}\mathcal{Y}_{r+1}$.*
(2) $\mathcal{V}_r\mathcal{V}_j = (p-1)p^{n-j-1}\mathcal{V}_r = \mathcal{V}_j\mathcal{V}_r$ where $r<j$.
(3) $\mathcal{W}_r^2 = \left(\frac{-1}{p}\right) p^{n-r-1}((p-1)\mathcal{Y}_{r+1} - \mathcal{V}_r)$.
(4) $\mathcal{W}_r\mathcal{W}_j = 0$ for every $r \neq j$.
(5) $\mathcal{V}_r\mathcal{W}_r = \mathcal{W}_r\mathcal{V}_r = -p^{n-r-1}\mathcal{W}_r$.
(6) $\mathcal{V}_r\mathcal{W}_j = \mathcal{W}_j\mathcal{V}_r = \begin{cases} 0 & \text{if } r<j \\ p^{n-r-1}(p-1)\mathcal{W}_j & \text{if } r>j \end{cases}$.
(7) $\mathcal{U}_0\mathcal{V}_r = p^{n-r-1}(p-1)\mathcal{U}_0 = \mathcal{V}_r\mathcal{U}_0$.
(8) $\mathcal{U}_0\mathcal{W}_r = 0 = \mathcal{W}_r\mathcal{U}_0$.
(9) For $\eta = 1$, $\mathcal{U}_0^2 = p^n\mathcal{Y}_1 + p^{n-1}(p-1)\mathcal{U}_0$; for $\eta = \left(\frac{\cdot}{p}\right)$, $\mathcal{U}_0^2 = \eta(-1)p^n\mathcal{Y}_1$.
(10) With $p^* = \left(\frac{-1}{p}\right)p$ as fixed in Section 2, the terminal operator satisfies $\mathcal{W}_{n-1}^3 = p^*\mathcal{W}_{n-1}$; equivalently, it satisfies $X(X^2-p^*) = 0$.

Indeed, at the terminal index $r = n-1$, we have $\mathcal{Y}_n = \mathcal{I}$ and $\mathcal{Y}_{n-1} = \mathcal{I} + \mathcal{V}_{n-1}$; hence parts (3) and (5) give

$$\mathcal{W}_{n-1}^2 = \left(\frac{-1}{p}\right)\big(p\mathcal{I} - \mathcal{Y}_{n-1}\big), \qquad \mathcal{Y}_{n-1}\mathcal{W}_{n-1} = 0.$$

Multiplying the first identity by $\mathcal{W}_{n-1}$ proves part (10).

Corollary 3.7. *The algebra $H(\overline{K}//\overline{K_n},\eta)$ is a $2n$-dimensional commutative algebra with generators $\mathcal{I},\mathcal{U}_0,\mathcal{V}_r,\mathcal{W}_r$ for $1 \le r \le n-1$, and relations given by Proposition 3.4 and Corollary 3.6.*

It follows from the above corollary that

$$\begin{aligned}
&\mathcal{Y}_r\mathcal{Y}_j = p^{n-j}\mathcal{Y}_r = \mathcal{Y}_j\mathcal{Y}_r, \quad \forall j \ge r,\\
&\mathcal{U}_0\mathcal{Y}_r = p^{n-r}\mathcal{U}_0 = \mathcal{Y}_r\mathcal{U}_0, \quad \forall r \in \{1,\dots,n-1\},\\
&\mathcal{Y}_r\mathcal{W}_j = \mathcal{W}_j\mathcal{Y}_r = \begin{cases} p^{n-r}\mathcal{W}_j, & \text{if } r>j,\\ 0, & \text{if } r \le j,\end{cases}\\
&\mathcal{W}_j\mathcal{W}_r = 0 \qquad \text{for any } r \neq j,\\
&\mathcal{W}_r\mathcal{U}_0 = \mathcal{U}_0\mathcal{W}_r = 0 \quad \forall r \in \{1,\dots,n-1\},\\
&\mathcal{W}_r^2 = \left(\frac{-1}{p}\right)p^{n-r-1}(p\mathcal{Y}_{r+1} - \mathcal{Y}_r),
\end{aligned}$$

$$\mathcal{U}_0^2 = p^n\mathcal{Y}_1 + p^{n-1}(p-1)\mathcal{U}_0 \quad \text{if } \eta = 1, \qquad \mathcal{U}_0^2 = \eta(-1)p^n\mathcal{Y}_1 \quad \text{if } \eta = \left(\frac{\cdot}{p}\right).$$

Remark 1. *The elements $\mathcal{Y}_r$ admit an averaging interpretation. For $1 \le r \le n$, the character η extends from $\overline{K_n}$ to $\overline{K_r}$ by the same formula as in Section 2, since its underlying character factors modulo p. The double cosets*

$\overline{K_n}$ and $\overline{K_n}(y(\delta p^j),1)\overline{K_n}$, with $r \le j \le n-1$ and $\delta \in \{1,\lambda\}$, partition $\overline{K_r}$. At each chosen representative, both η and the corresponding normalized Hecke function have value 1. Biequivariance and the multiplicativity of η on $\overline{K_r}$ therefore give

$$\mathcal{Y}_r(\tilde{g}) = \begin{cases} \eta(\tilde{g})^{-1}, & \tilde{g} \in \overline{K_r}, \\ 0, & \tilde{g} \notin \overline{K_r}. \end{cases}$$

With $\mathrm{vol}(\overline{K_n}) = 1$, *we have* $\mathrm{vol}(\overline{K_r}) = [K_r : K_n] = p^{n-r}$. *Thus, on the $(\overline{K_n},\eta)$-fixed space of a smooth genuine representation (π, V), the operator*

$$p^{r-n}\pi(\mathcal{Y}_r)v = \frac{1}{\mathrm{vol}(\overline{K_r})}\int_{\overline{K_r}} \eta(\tilde{k})^{-1}\pi(\tilde{k})v\,d\tilde{k}$$

is the projection onto the $(\overline{K_r},\eta)$-fixed space.

3.1. **Representations of $\overline{K}$ having a vector of prescribed $\overline{K_n}$-type.** In [1], we studied representations of $\mathrm{GL}_2(\mathbb{Z}_p)$ containing an $H_0(p^n)$-fixed vector, where $H_0(p^n)$ consists of matrices in $\mathrm{GL}_2(\mathbb{Z}_p)$ which reduce to upper-triangular matrices modulo $p^n\mathbb{Z}_p$.

We would like to study the irreducible representations of $\overline{K}$ having a nonzero vector of $(\overline{K_n},\overline{\eta})$-type. Let

$$I(n) := Ind_{\overline{K_n}}^{\overline{K}}\overline{\eta} = \left\{ \phi : \overline{K} \to \mathbb{C} \;\middle|\; \begin{array}{l} \phi(k_0k) = \overline{\eta}(k_0)\phi(k), \\ \forall k_0 \in \overline{K_n},\ k \in \overline{K} \end{array} \right\}.$$

ϕ's are smooth (locally constant). Then $I(n)$ is a right representation of $\overline{K}$ via right translation, and the dimension of this representation is $[\overline{K} : \overline{K_n}] = p^{n-1}(p+1)$.

The set

$$I(n)_{\overline{\eta}}^{K_n} = \{\phi \in I(n) : \phi(kk_0) = \overline{\eta}(k_0)\phi(k), \\ \forall k_0 \in \overline{K_n},\ k \in \overline{K}\}$$

is the collection of vectors of $(\overline{K_n},\overline{\eta})$-type in $I(n)$ and hence equals the Hecke algebra H. Consequently $\dim I(n)_{\overline{\eta}}^{K_n} = 2n$.

By Frobenius reciprocity,

$$\begin{aligned}\mathrm{Hom}_{\overline{K}}(I(n), I(n)) &\cong \mathrm{Hom}_{\overline{K_n}}(\overline{\eta}, I(n)|_{\overline{K_n}}) \\ &\cong I(n)_{\overline{\eta}}^{K_n} \cong H.\end{aligned}$$

Since $\overline{K}$ is compact, $I(n)$ is completely reducible. Moreover, $H \cong \mathrm{End}_{\overline{K}}(I(n))$ is commutative and has dimension $2n$. It follows that $I(n)$ is a multiplicity-free direct sum of $2n$ irreducible representations. Any irreducible smooth representation of $\overline{K}$ containing a nonzero vector of $(\overline{K_n},\overline{\eta})$-type is isomorphic to a subrepresentation of $I(n)$. We describe these subrepresentations and their exact-level type vectors.

Corollary 3.8. *Let $n > 1$. There exist exactly two irreducible constituents $\sigma(n), \sigma'(n)$ of $I(n)$ that do not occur in $I(n-1)$. Each contains, up to scalar multiple, a unique vector of $(\overline{K_n},\overline{\eta})$-type and no nonzero vector of $(\overline{K_{n-1}},\overline{\eta})$-type. At level 1, the two irreducible constituents of $I(1)$ have dimensions summing to $p+1$, while for $n \ge 2$ one has* $\dim(\sigma(n)) + \dim(\sigma'(n)) = p^{n-2}(p^2-1)$.

If we restrict ourselves to $\eta \equiv 1$, then let S be the subalgebra of H generated by $\mathcal{I}, \mathcal{U}_0, \mathcal{Y}_r$ for every $r \in \{1, \ldots, n-1\}$. These elements satisfy exactly the same relations as elements of the subalgebra $H(\mathrm{GL}_2(\mathbb{Z}_p)//H_0(p^n))$ and hence are isomorphic. The algebra H is generated by S together with $\mathcal{W}_1, \ldots, \mathcal{W}_{n-1}$.

We have a left action π_L of H on $I(n)$ given by $\pi_L(f)(\phi) = f * \phi$ for $f \in H$, $\phi \in I(n)$, in fact $\pi_L(f) \in \mathrm{Hom}_{\overline{K}}(I(n), I(n))$ and so acts by scalar on irreducible components of $I(n)$.

With the notation $p^* = \left(\frac{-1}{p}\right) p$ introduced above, as in [1, Proposition 3.17], a basis of eigenvectors of H under π_L consists of

$$\begin{array}{cc} \eta \equiv 1 & \eta \not\equiv 1 \\ u_1 = \mathcal{U}_0 + \mathcal{Y}_1 & u_1 = \mathcal{U}_0 + \sqrt{p^*}\mathcal{Y}_1 \\ u_2 = \mathcal{U}_0 - p\mathcal{Y}_1 & u_2 = \mathcal{U}_0 - \sqrt{p^*}\mathcal{Y}_1 \end{array}$$

and, in both cases,

$$\begin{aligned} v_r &= p\mathcal{Y}_{r+1} - \mathcal{Y}_r + \frac{p}{\sqrt{p^*}}\mathcal{W}_r, \\ v_r' &= p\mathcal{Y}_{r+1} - \mathcal{Y}_r - \frac{p}{\sqrt{p^*}}\mathcal{W}_r, \qquad 1 \le r \le n-1. \end{aligned}$$

For $\eta \equiv 1$, the eigenvalues are as follows:

	$\mathcal{U}_0$	$\mathcal{Y}_1$	$\mathcal{Y}_2$	$\ldots$	$\mathcal{Y}_{n-1}$	$\mathcal{W}_1$	$\mathcal{W}_2$	$\ldots$	$\mathcal{W}_{n-1}$
u_1	p^n	p^{n-1}	p^{n-2}	$\ldots$	p	0	0	$\ldots$	0
u_2	$-p^{n-1}$	p^{n-1}	p^{n-2}	$\ldots$	p	0	0	$\ldots$	0
v_1	0	0	p^{n-2}	$\ldots$	p	$p^{n-2}\sqrt{p^*}$	0	$\ldots$	0
v_1'	0	0	p^{n-2}	$\ldots$	p	$-p^{n-2}\sqrt{p^*}$	0	$\ldots$	0
v_2	0	0	0	$\ldots$	p	0	$p^{n-3}\sqrt{p^*}$	$\ldots$	0
v_2'	0	0	0	$\ldots$	p	0	$-p^{n-3}\sqrt{p^*}$	$\ldots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{n-1}	0	0	0	$\ldots$	0	0	0	$\ldots$	$\sqrt{p^*}$
v_{n-1}'	0	0	0	$\ldots$	0	0	0	$\ldots$	$-\sqrt{p^*}$

For $\eta \not\equiv 1$, all entries remain the same except that the $\mathcal{U}_0$-eigenvalues on u_1, u_2 are $p^{n-1}\sqrt{p^*}$ and $-p^{n-1}\sqrt{p^*}$, respectively.

Lemma 3.9. *The operators $\pi_L(\mathcal{U}_0), \pi_L(\mathcal{V}_r)$ and $\pi_L(\mathcal{W}_r)$ have zero trace.*

Proof. Choose representatives $(g, 1)$ for the left cosets of $\overline{K_n}$ in $\overline{K}$. Let ϕ_g be supported on $\overline{K_n}(g, 1)$ and normalized by

$$\phi_g(\tilde{k}(g, 1)) = \overline{\eta}(\tilde{k}), \qquad \tilde{k} \in \overline{K_n}.$$

These functions form a basis of $I(n)$. Since the supporting double coset of each of $\mathcal{U}_0, \mathcal{V}_{r,1}, \mathcal{V}_{r,\lambda}$ is disjoint from $\overline{K_n}$, a direct support argument gives

$$\pi_L(\mathcal{V}_{r,1})(\phi_g)((g, 1)) = \pi_L(\mathcal{V}_{r,\lambda})(\phi_g)((g, 1)) = \pi_L(\mathcal{U}_0)(\phi_g)((g, 1)) = 0.$$

Thus $\pi_L(\mathcal{U}_0)$, $\pi_L(\mathcal{V}_{r,1})$ and $\pi_L(\mathcal{V}_{r,\lambda})$ are traceless. By linearity of trace, so are $\pi_L(\mathcal{V}_r)$ and $\pi_L(\mathcal{W}_r)$. ☐

For each of the displayed basis vectors v, put $d_v = \dim \mathrm{Span}(\pi_R(\overline{K})v)$. By Lemma 3.9,

$$0 = \mathrm{tr}\, \pi_L(\mathcal{W}_r) = p^{n-r-1}\sqrt{p^*}(d_{v_r} - d_{v_r'}),$$

so $d_{v_r} = d_{v'_r}$ for $1 \le r \le n-1$. The $\mathcal{U}_0$-trace equation gives

$$d_{u_2} = \begin{cases} p\, d_{u_1}, & \eta \equiv 1, \\ d_{u_1}, & \eta \not\equiv 1. \end{cases}$$

The $\mathcal{V}_r$-eigenvalues are identical for both characters. After canceling the common factor p^{n-r-1}, their trace equations give

$$(p-1)\left(d_{u_1} + d_{u_2} + \sum_{j=1}^{r-1}(d_{v_j} + d_{v'_j})\right) = d_{v_r} + d_{v'_r}, \qquad 1 \le r \le n-1.$$

Together with

$$d_{u_1} + d_{u_2} + \sum_{r=1}^{n-1}(d_{v_r} + d_{v'_r}) = \dim I(n) = p^{n-1}(p+1),$$

these equations give the following dimensions.

Corollary 3.10. *The representation $I(n)$ is the multiplicity-free sum of the $2n$ irreducible representations*

$$\begin{aligned} S_i &= \operatorname{Span}(\pi_R(\overline{K})u_i), & i &= 1, 2, \\ T_r &= \operatorname{Span}(\pi_R(\overline{K})v_r), & T'_r &= \operatorname{Span}(\pi_R(\overline{K})v'_r), \qquad 1 \le r \le n-1. \end{aligned}$$

Their dimensions are

$$(\dim S_1, \dim S_2) = \begin{cases} (1, p), & \eta \equiv 1, \\ \left(\frac{p+1}{2}, \frac{p+1}{2}\right), & \eta \not\equiv 1, \end{cases} \quad \dim T_r = \dim T'_r = p^{r-1}\frac{p^2-1}{2}.$$

4. Local Newforms in Half Integral Weight

4.1. **Induced Representations.** Retain the additive character ψ of conductor 0 fixed in Section 2. Let μ be a character of $\mathbb{Q}_p^\times$ of conductor exponent n. We first define explicitly the genuine character used in the induction. For $a \in \mathbb{Q}_p^\times$, $b \in \mathbb{Q}_p$, and $\epsilon \in \{\pm 1\}$, put

$$\chi_\psi((h(a)x(b), \epsilon)) = \epsilon\gamma(a, \psi)^{-1}.$$

The restriction of our cocycle to the torus is $c(h(a), h(a')) = (a, a')_p$. Hence the identity $\gamma(aa', \psi) = \gamma(a, \psi)\gamma(a', \psi)(a, a')_p$, shows that χ_ψ is a genuine character of B; it is trivial on the canonical lift of the upper unipotent subgroup. Define

$$\widetilde{\mu}_\psi((h(a)x(b), \epsilon)) = \mu(a)\chi_\psi((h(a)x(b), \epsilon)) = \epsilon\mu(a)\gamma(a, \psi)^{-1}.$$

We use normalized induction and set

$$\pi_\psi(\mu) = \operatorname{Ind}_B^{\widetilde{G}} \widetilde{\mu}_\psi,$$

with representation space $V(\mu)$. Thus $V(\mu)$ consists of the locally constant functions $f : \widetilde{G} \to \mathbb{C}$ satisfying

$$f((h(a)x(b), \epsilon)g) = |a|\, \widetilde{\mu}_\psi((h(a)x(b), \epsilon))f(g) = \epsilon|a|\mu(a)\gamma(a, \psi)^{-1}f(g).$$

This is the standard genuine torus character and normalized induced model used by Baruch–Mao, written there with a different Weil-index notation; see [15]. It also agrees with the normalization in [16, §3.2]. Baruch–Mao and Ishimoto formulate the construction in the σ_0-model. Since $s_p(g) = 1$ for every $g \in B_0$, the isomorphism ι is the identity on the full inverse image of B_0. Therefore the genuine character, the factor $|a|$ coming from normalized induction, and the displayed transformation law are unchanged in our c-model. For brevity, write

$$\mu_B((h(a)x(b), \epsilon)) := |a|\, \widetilde{\mu}_\psi((h(a)x(b), \epsilon)).$$

Since $c(\psi) = 0$, the Weil-factor identities in Section 2 give $\gamma(u, \psi) = 1$ for $u \in \mathbb{Z}_p^\times$. Consequently, on the compact diagonal elements used in the support calculations below, $\mu_B((h(u), \epsilon)) = \epsilon\mu(u)$.

Now let η be either the trivial character or the character $\eta = \left(\frac{\cdot}{p}\right)$, with the conventions of Section 2. For $m \geq c(\eta)$, we write

$$\pi_\psi(\mu)_\eta^{K_m} = V(\mu)_\eta^{K_m} = V_\eta^{K_m}\big(\pi_\psi(\mu)\big).$$

In the induced model this space is

$$V(\mu)_\eta^{K_m} = \{f \in V(\mu) : f(g\tilde{k}) = \eta(\tilde{k})f(g) \text{ for all } g \in \widetilde{G},\ \tilde{k} \in \overline{K_m}\}.$$

Recall that $c_\eta(\pi_\psi(\mu))$ is the least such m for which this space is nonzero.

For each m, the Hecke subalgebra $H_m(\eta) = H(\overline{K}//\overline{K_m}, \eta)$ of Section 2 acts on $V(\mu)_\eta^{K_m}$. We write $H_m = H_m(\eta)$ when the character is understood. We shall determine the required simultaneous eigenspaces explicitly. Fix again a quadratic non-residue $\lambda \in \mathbb{Z}_p$ as in the previous section.

Lemma 4.1. *The double cosets*

$$B\overline{K_m}, \qquad B(w(1), 1)\overline{K_m},$$
$$B(y(p^r), 1)\overline{K_m}, \qquad B(y(\lambda p^r), 1)\overline{K_m}$$

where $r \in \{1, \ldots, m-1\}$ and λ is a nonsquare modulo p, form a complete set of double coset representatives for $B\backslash\widetilde{G}/\overline{K_m}$.

Proof. The fact that these cosets cover $\widetilde{G}$ follows from the Iwasawa decomposition $\widetilde{G} = B\overline{K}$ and Proposition 2.1.

To show disjointness, assume $y(t_1)k_0y(-t_2) \in B_0$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = k_0 \in K_m$. A direct calculation gives

$$t_2a + c - t_1t_2b - t_1d = 0.$$

Write $t_i = \ell_i p^{r_i}$ with $\ell_i \in \mathbb{Z}_p^\times$ and $1 \leq r_i < m$. If $r_1 \neq r_2$, the term having the smaller valuation cannot be cancelled by any of the other three terms. Hence $r_1 = r_2 = r$. After division by p^r and reduction modulo p, the determinant condition gives

$$\ell_2 a - \ell_1 a^{-1} \equiv 0 \pmod{p}.$$

Thus ℓ_1 and ℓ_2 have the same square class modulo p. The remaining comparisons, involving 1, $w(1)$, and $y(t)$, follow by the same lower-row calculation. This proves disjointness. □

Throughout the remainder of this subsection, we assume the central compatibility condition $\mu(-1) = \eta(-1)$. This is necessary for a nonzero $(\overline{K_m}, \eta)$-fixed vector, since $(h(-1), 1)$ acts on $V(\mu)$ by the scalar $\mu(-1)$.

Proposition 4.2. *Assume $n > 1$. Then $V(\mu)_\eta^{K_{2n}}$ is the 2-dimensional space spanned by the functions*

$$f_1(g) = \begin{cases} \mu_B(b)\eta(k_0) & \text{if } g = b(y(p^n), 1)k_0, \text{ for some } b \in B, k_0 \in \overline{K_{2n}} \\ 0 & \text{otherwise} \end{cases}$$

$$f_\lambda(g) = \begin{cases} \mu_B(b)\eta(k_0) & \text{if } g = b(y(\lambda p^n), 1)k_0, \text{ for some } b \in B, k_0 \in \overline{K_{2n}} \\ 0 & \text{otherwise} \end{cases}$$

Consequently $c_\eta(\pi_\psi(\mu)) = 2n$.

Proof. We give the support argument at an arbitrary level K_M. By Lemma 4.1, a vector in $V(\mu)_\eta^{K_M}$ is a sum of functions supported on the displayed B–K_M double cosets. A nonzero function supported on $Bg\overline{K_M}$ exists precisely when the inducing character and the right η-type agree on $B \cap g\overline{K_M}g^{-1}$. Since all the representatives lie in K and the cover splits over K, no additional cocycle factor occurs in this check.

The cosets $B\overline{K_M}$ and $B(w(1), 1)\overline{K_M}$ cannot support such a function. Indeed, applying the compatibility condition to the diagonal elements $h(u)$, $u \in \mathbb{Z}_p^\times$, would give respectively $\mu|_{\mathbb{Z}_p^\times} = \eta$ or $\mu^{-1}|_{\mathbb{Z}_p^\times} = \eta$. Either equality contradicts $c(\mu) = n > 1$, since $c(\eta) \le 1$.

It remains to examine $g = (y(\delta p^k), 1)$, where $\delta \in \{1, \lambda\}$ and $1 \le k < M$. Writing $b = (\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}, 1) \in B$, the condition $g^{-1}bg \in \overline{K_M}$ is equivalent, on the linear group, to

$$\begin{pmatrix} \alpha + \beta\delta p^k & \beta \\ p^k(\alpha^{-1} - \alpha - \beta\delta p^k) & \alpha^{-1} - \beta\delta p^k \end{pmatrix} \in K_M.$$

Thus $\beta \in \mathbb{Z}_p$, $\alpha \in \mathbb{Z}_p^\times$, and

$$\alpha^{-1} - \alpha - \beta\delta p^k \equiv 0 \pmod{p^{M-k}}.$$

For such an element the support compatibility is

$$\mu(\alpha^{-1}) = \eta(\alpha^{-1} - \beta\delta p^k).$$

If $k < n$, choose $\alpha \in 1 + p^k\mathbb{Z}_p$ and $\beta = (\alpha^{-1} - \alpha)/(\delta p^k)$. The support compatibility then forces μ to be trivial on $1 + p^k\mathbb{Z}_p$, contrary to $c(\mu) = n$. If $M - k < n$, take $\beta = 0$ and $\alpha \in 1 + p^{M-k}\mathbb{Z}_p$; the same compatibility forces μ to be trivial on $1 + p^{M-k}\mathbb{Z}_p$, again a contradiction. Therefore an y-coset supporting a nonzero type function must satisfy

$$k \ge n, \qquad M - k \ge n.$$

In particular, no such coset occurs when $M < 2n$, proving $V(\mu)_\eta^{K_M} = 0$ for every $M < 2n$.

At $M = 2n$, these two inequalities force $k = n$. In this case the preceding congruence gives $\alpha^2 \equiv 1 \pmod{p^n}$. Since p is odd, this means $\alpha \in \pm(1 + p^n\mathbb{Z}_p)$. As μ is trivial on $1 + p^n\mathbb{Z}_p$ and $c(\eta) \le 1$, the support compatibility is equivalent to $\mu(-1) = \eta(-1)$. Hence precisely the two cosets represented by $y(p^n)$ and $y(\lambda p^n)$ support nonzero type functions, namely f_1 and f_λ. They are linearly independent, and each supporting space is one-dimensional. This proves both the dimension assertion and $c_\eta(\pi_\psi(\mu)) = 2n$. □

Remark 2. *Ishimoto independently obtained the same conductor and dimension statement in* [16, Theorem 4.1]*, using Frobenius reciprocity and Mackey decomposition. He also describes the corresponding functions supported on the two square-class double cosets in* [16, Corollary 4.3(4)]*; in the notation used here, these are the two support lines generated by f_1 and f_λ. The proof above is a direct support computation.*

Remark 3. *Suppose that $n = 1$. In the cancelling case $\eta = \mu^{\pm 1}|_{\mathbb{Z}_p^\times}$, the quadraticity of η implies that both $\mu|_{\mathbb{Z}_p^\times}$ and $\mu^{-1}|_{\mathbb{Z}_p^\times}$ equal η. Consequently $V(\mu)_\eta^{K_1}$ is two-dimensional, spanned by the standard type functions supported on $B\overline{K}_1$ and $B(w(1),1)\overline{K}_1$, and $c_\eta(\pi_\psi(\mu)) = 1$. In the remaining case, $c(\eta\mu) > 0$ and $c(\eta\mu^{-1}) > 0$; then $V(\mu)_\eta^{K_1} = 0$, while the two functions supported on $B(y(p),1)\overline{K}_2$ and $B(y(\lambda p),1)\overline{K}_2$ span $V(\mu)_\eta^{K_2}$. Thus the η-conductor is 2, and this case is treated in Proposition 4.5; compare* [16, Theorem 4.1 and Corollary 4.3].

We will now use the results on eigenvalues from the previous section to identify the eigenvalues of the newforms inside this two-dimensional space. On $\mathbb{Z}_p^\times$ we write $\overline{\mu} = \mu^{-1}$; since the restriction of μ to $\mathbb{Z}_p^\times$ has finite order, this is also its complex conjugate.

Proposition 4.3. *Let $\bar{g} = (g, 1) \in \widetilde{G}$. For $n > 1$ we obtain the following equalities:*

$$\mathcal{V}_{2n-1,\kappa}(f_\kappa)(\bar{y}(\delta p^n)) = \begin{cases} \frac{-1\pm\sqrt{p^*}}{2} & \text{if } \delta = \kappa \\ 0 & \text{if } \delta \neq \kappa \end{cases}$$

$$\mathcal{V}_{2n-1,\beta}(f_\kappa)(\bar{y}(\delta p^n)) = \begin{cases} \frac{-1\mp\sqrt{p^*}}{2} & \text{if } \delta = \kappa \\ 0 & \text{if } \delta \neq \kappa \end{cases}$$

where $\beta \neq \kappa$ in the second case and $\beta, \kappa, \delta \in \{1, \lambda\}$. The signs depend on μ.

Proof. For $\beta \in \{1, \lambda\}$: $\mathcal{V}_{2n-1,\beta}(f)(g) = \sum_{u \in \mathbb{Z}_p^\times / \sqrt{1+p\mathbb{Z}_p}} f(g\bar{h}(u)\bar{y}(\beta p^{2n-1}))\eta(u)$.

Now taking $g = \bar{y}(\delta p^n)$ and $f = f_\kappa$ the argument in the summation becomes

$$\begin{pmatrix} 1 & 0 \\ \delta p^n & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta p^{2n-1} & 1 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \delta p^n u^2 + \beta p^{2n-1} & 1 \end{pmatrix}.$$

From a straightforward calculation it follows that there exists a matrix $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in B_0$ such that $\begin{pmatrix} 1 & 0 \\ -\kappa p^n & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \delta p^n u^2 + \beta p^{2n-1} & 1 \end{pmatrix} \in K_{2n}$ if and only if $\delta u^2 + \beta p^{n-1} \equiv \kappa a^2 \pmod{p^n}, b \in \mathbb{Z}_p$ and $a \in \mathbb{Z}_p^\times$.

This relation has a solution if and only if $\delta = \kappa$. In this case we choose the square root by setting

$$a = u\sqrt{1 + \frac{\beta}{\kappa} p^{n-1} u^{-2}}, \qquad \sqrt{1 + \frac{\beta}{\kappa} p^{n-1} u^{-2}} \in 1 + p^{n-1}\mathbb{Z}_p.$$

This is the unique choice for which $a \equiv u \pmod{p}$. Since $c(\eta) \leq 1$, it follows that $\eta(a) = \eta(u)$.

First let $\kappa = \delta = \beta$, from which we obtain

$$\mathcal{V}_{2n-1,\kappa}(f_\kappa)(\bar{y}(\kappa p^n)) = \sum_{u\in\mathbb{Z}_p^\times/\sqrt{1+p\mathbb{Z}_p}} \bar{\mu}(\frac{\sqrt{u^2+p^{n-1}}}{u})\eta(\sqrt{u^2+p^{n-1}})\eta(u).$$

With the choice just made, $\eta(\sqrt{u^2+p^{n-1}}) = \eta(u)$ and $\frac{\sqrt{u^2+p^{n-1}}}{u} = \sqrt{1+p^{n-1}u^{-2}} = 1+\frac{1}{2}p^{n-1}u^{-2} \pmod{p^n}$, so the last sum evaluates to $\sum_{u\in\mathbb{Z}_p^\times/\sqrt{1+p\mathbb{Z}_p}} \bar{\mu}(1+\frac{1}{2}p^{n-1}u^{-2})$.

Since $c(\mu) = n > 1$, the map $t \longmapsto \overline{\mu}\big(1+\frac{1}{2}p^{n-1}t\big)$ defines a nontrivial additive character of $\mathbb{F}_p$. Put $\zeta = \overline{\mu}\big(1+\frac{1}{2}p^{n-1}\big)$. Its quadratic Gauss sum, in the two equivalent forms of Section 2, is

$$g_\zeta(1;p) := \sum_{x\in\mathbb{F}_p} \zeta^{x^2} = \sum_{t\in\mathbb{F}_p^*} \chi_p(t)\zeta^t = \pm\sqrt{p^*}.$$

The sign is determined by μ, relative to the choice of $\sqrt{p^*}$ fixed in Section 2. Reindexing by $u \mapsto u^{-1}$, which permutes the quotient, gives

$$\begin{aligned}\mathcal{V}_{2n-1,\kappa}(f_\kappa)(\bar{y}(\kappa p^n)) &= \sum_{u\in\mathbb{Z}_p^\times/\sqrt{1+p\mathbb{Z}_p}} \zeta^{u^2} = \frac{1}{2}\sum_{u\in\mathbb{Z}_p^\times/(1+p\mathbb{Z}_p)} \zeta^{u^2}\\ &= \frac{g_\zeta(1;p)-1}{2} = \frac{-1\pm\sqrt{p^*}}{2}.\end{aligned}$$

In the same way for $\beta \neq \kappa = \delta$ we obtain

$$\mathcal{V}_{2n-1,\beta}(f_\kappa)(\bar{y}(\kappa p^n)) = \frac{1}{2}\sum_{u\in\mathbb{Z}_p^\times/(1+p\mathbb{Z}_p)} \zeta^{\frac{\beta}{\kappa}u^2} = \frac{-1-g_\zeta(1;p)}{2} = \frac{-1\mp\sqrt{p^*}}{2}.$$

Notice that in all of our calculations we remain inside K and $\widetilde{G}$ splits over K, so the cocycle component plays no part in the computation. $\square$

Corollary 4.4. *The functions f_1, f_λ defined in Proposition 4.2 are the common eigenvectors in $V(\mu)_\eta^{K_{2n}}$ of the operators $\mathcal{Y}_{2n-1}$ and $\mathcal{W}_{2n-1}$ from the previous section. The forms and their corresponding eigenvalues are given in the table below.*

Proof. Proposition 4.3 shows directly that each of f_1 and f_λ is an eigenvector for both $\mathcal{V}_{2n-1,1}$ and $\mathcal{V}_{2n-1,\lambda}$. Since at the terminal index

$$\mathcal{Y}_{2n-1} = I + \mathcal{V}_{2n-1,1} + \mathcal{V}_{2n-1,\lambda}, \qquad \mathcal{W}_{2n-1} = \mathcal{V}_{2n-1,1} - \mathcal{V}_{2n-1,\lambda},$$

the displayed eigenvalues in Proposition 4.3 give $\mathcal{Y}_{2n-1}f_\kappa = 0$ and $\mathcal{W}_{2n-1}f_\kappa = \pm\sqrt{p^*}\, f_\kappa$, with opposite signs for $\kappa = 1$ and λ. $\square$

Case 1	f_1	f_λ
$\mathcal{V}_{2n-1,1}$	$\frac{-1\pm\sqrt{p^*}}{2}$	$\frac{-1\mp\sqrt{p^*}}{2}$
$\mathcal{V}_{2n-1,\lambda}$	$\frac{-1\mp\sqrt{p^*}}{2}$	$\frac{-1\pm\sqrt{p^*}}{2}$

Case 2	f_1	f_λ
$\mathcal{V}_{2n-1,1}$	$\frac{-1\mp\sqrt{p^*}}{2}$	$\frac{-1\pm\sqrt{p^*}}{2}$
$\mathcal{V}_{2n-1,\lambda}$	$\frac{-1\pm\sqrt{p^*}}{2}$	$\frac{-1\mp\sqrt{p^*}}{2}$

Case 1	f_1	f_λ
$\mathcal{Y}_{2n-1}$	0	0
$\mathcal{W}_{2n-1}$	$\sqrt{p^*}$	$-\sqrt{p^*}$

Case 2	f_1	f_λ
$\mathcal{Y}_{2n-1}$	0	0
$\mathcal{W}_{2n-1}$	$-\sqrt{p^*}$	$\sqrt{p^*}$

Proposition 4.5. *Let $n = 1$ and assume $c(\eta\mu) > 0$ and $c(\eta\mu^{-1}) > 0$, so that $c_\eta(\pi_\psi(\mu)) = 2$. Then the 2-dimensional space spanned by the functions*

$$f_1(g) = \begin{cases} \mu_B(b)\eta(k_0) & \text{if } g = b(y(p),1)k_0, \text{ for some } b \in B, k_0 \in \overline{K_2} \\ 0 & \text{otherwise} \end{cases}$$

$$f_\lambda(g) = \begin{cases} \mu_B(b)\eta(k_0) & \text{if } g = b(y(\lambda p),1)k_0, \text{ for some } b \in B, k_0 \in \overline{K_2} \\ 0 & \text{otherwise} \end{cases}$$

has a basis $\{v_1, v_2\}$ such that $\mathcal{U}_0(v_1) = \mathcal{U}_0(v_2) = \mathcal{Y}_1(v_1) = \mathcal{Y}_1(v_2) = 0$ and $\mathcal{W}_1(v_1) = \sqrt{p^*}v_1, \mathcal{W}_1(v_2) = -\sqrt{p^*}v_2$.

Proof. We have that

$$\mathcal{U}_0(f_\kappa)(\bar{y}(\delta p)) = \sum_{s \in \mathbb{Z}_p/p^2\mathbb{Z}_p} f_\kappa(\bar{y}(\delta p)\bar{x}(s)\bar{w}(1)).$$

Let

$$M = \begin{pmatrix} -s & 1 \\ -1-\delta ps & \delta p \end{pmatrix},$$

then the sum is zero unless $Mk_0y(-\kappa p) \in B_0$ for some $k_0 \in K_2$. But

$$\begin{aligned} &\begin{pmatrix} -s & 1 \\ -1-\delta ps & \delta p \end{pmatrix} \begin{pmatrix} a & b \\ p^2c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\kappa p & 1 \end{pmatrix} \\ &= \begin{pmatrix} -s & 1 \\ -1-\delta ps & \delta p \end{pmatrix} \begin{pmatrix} a-\kappa pb & b \\ p^2c-\kappa pd & d \end{pmatrix} \\ &= \begin{pmatrix} * & * \\ (-1-\delta ps)(a-\kappa pb)+\delta p(p^2c-\kappa pd) & * \end{pmatrix} \notin B_0 \end{aligned}$$

since $a \in \mathbb{Z}_p^\times$. It follows that $\mathcal{U}_0(f_\kappa)(\bar{y}(\delta p)) = 0$ for every $\delta, \kappa \in \{1, \lambda\}$. Since $\mathcal{U}_0(f_\kappa)$ belongs to the space spanned by f_1, f_λ, and these two values determine a vector in that space, $\mathcal{U}_0$ acts by zero on all of $V(\mu)_\eta^{K_2}$.

Since $c_\eta(\pi_\psi(\mu)) = 2$, the space $V(\mu)_\eta^{K_1}$ is zero. At level 2, Remark 1 shows that $\mathcal{Y}_1 = \mathcal{I} + \mathcal{V}_{1,1} + \mathcal{V}_{1,\lambda}$ is supported on $\overline{K_1}$ and takes the value $\eta(\tilde{k})^{-1}$ at each $\tilde{k} \in \overline{K_1}$. Hence $p^{-1}\mathcal{Y}_1$ acts as the averaging projection onto $V(\mu)_\eta^{K_1}$, and $\mathcal{Y}_1$ acts by zero on $V(\mu)_\eta^{K_2}$.

The relation $\mathcal{W}_1^2 = p^*I$ on this space shows that $\mathcal{W}_1$ is diagonalizable and that its eigenvalues belong to $\{\sqrt{p^*}, -\sqrt{p^*}\}$. We show that both occur. From the same calculations as in Proposition 4.3 for $n = 1$ we obtain

$$\mathcal{V}_{1,1}(f_1)(\bar{y}(p)) = \mathcal{V}_{1,\lambda}(f_\lambda)(\bar{y}(\lambda p)) = \sum_{\substack{u \in \mathbb{Z}_p^\times/\sqrt{1+p\mathbb{Z}_p} \\ \left(\frac{u^2+1}{p}\right)=1}} \bar{\mu}\left(\frac{\sqrt{u^2+1}}{u}\right)\eta(u\sqrt{u^2+1}) \tag{*}$$

Assume that only one $\mathcal{W}_1$-eigenvalue occurs. Since $\mathcal{W}_1$ is diagonalizable, it is then scalar. As $\mathcal{Y}_1 = 0$, both $\mathcal{V}_{1,1}$ and $\mathcal{V}_{1,\lambda}$ are scalar as well. We write their scalar values in the following table:

	f_1	f_λ
$\mathcal{V}_{1,1}$	$\ell_{1,1}$	$\ell_{1,\lambda}$
$\mathcal{V}_{1,\lambda}$	$\ell_{\lambda,1}$	$\ell_{\lambda,\lambda}$

Relation (*) gives $\ell_{1,1} = \ell_{\lambda,\lambda}$. Since the two operators are scalar, we also have $\ell_{1,1} = \ell_{1,\lambda}$ and $\ell_{\lambda,1} = \ell_{\lambda,\lambda}$. Hence $\mathcal{V}_{1,1}$ and $\mathcal{V}_{1,\lambda}$ act by the same scalar, so $\mathcal{W}_1 = 0$, contradicting $\mathcal{W}_1^2 = p^* I$. Both eigenvalues therefore occur. Choosing corresponding eigenvectors v_1, v_2 proves the proposition. □

4.2. **Reducible Principal Series: Even Weil and Steinberg Representations.** We study the two irreducible constituents of the reducible principal series in Proposition 4.6. We first recall the Weil representation in the conventions fixed in Section 2; compare [16, §3.2].

$$[\omega_\psi((h(a),\epsilon))\cdot\varphi](y) = \epsilon|a|^{\frac{1}{2}}\gamma(a,\psi)^{-1}\varphi(ay),$$
$$[\omega_\psi((x(b),1))\cdot\varphi](y) = \psi(by^2)\varphi(y),$$
$$[\omega_\psi((w(1),1))\cdot\varphi](y) = \gamma(\psi)\int_{\mathbb{Q}_p}\varphi(x)\psi(2xy)d_{\psi_2}x,$$

Here $d_{\psi_2}x$ is self-dual for the character $x \mapsto \psi(2x)$. Since p is odd and $c(\psi) = 0$, we take $\mathrm{vol}_{d_{\psi_2}x}(\mathbb{Z}_p) = 1$. Let $\mathcal{S}^+(\mathbb{Q}_p)$ and $\mathcal{S}^-(\mathbb{Q}_p)$ denote the subspaces of even and odd functions, respectively.

Thus

$$\mathcal{S}(\mathbb{Q}_p) = \mathcal{S}^+(\mathbb{Q}_p)\oplus\mathcal{S}^-(\mathbb{Q}_p), \qquad \omega_\psi = \omega_\psi^+\oplus\omega_\psi^-.$$

The even and odd Weil representations are irreducible; the former is non-supercuspidal and the latter is supercuspidal. For $a \in \mathbb{Q}_p^\times$, put $\mu_a(x) = (a,x)_p$, and recall $\psi_a(x) = \psi(ax)$. The isomorphism class of ω_{ψ_a} depends only on the square class of a, and we write

$$\omega^\pm_{\psi,\mu_a} := \omega^\pm_{\psi_a}.$$

See [16, §3.2] for these facts.

The reducibility properties of these representations are summarized in [16, Proposition 3.3] as follows.

Proposition 4.6. *(1) The representation $\pi_\psi(\mu)$ is irreducible if and only if $\mu^2 \neq |\cdot|^{\pm 1}$. In this case we have $\pi_\psi(\mu) \cong \pi_\psi(\mu^{-1})$.*

(2) If $\mu = \mu_a|\cdot|^{\frac{1}{2}}$, where μ_a is a quadratic or trivial character, then $\pi_\psi(\mu)$ is reducible and there are an irreducible representation St_{ψ,μ_a} and a short exact sequence

$$0 \to St_{\psi,\mu_a} \to \pi_\psi(\mu) \to \omega^+_{\psi,\mu_a} \to 0.$$

We shall call the representation St_{ψ,μ_a} the Steinberg representation associated to ψ and μ_a.

(3) If $\mu = \mu_a|\cdot|^{-\frac{1}{2}}$, where μ_a is a quadratic or trivial character, then $\pi_\psi(\mu)$ is reducible and there is a short exact sequence

$$0 \to \omega^+_{\psi,\mu_a} \to \pi_\psi(\mu) \to St_{\psi,\mu_a} \to 0.$$

The proposition gives all the irreducible non-supercuspidal genuine representations of $\widetilde{G}$; see [16, Proposition 3.3].

Choose a nonsquare unit $\beta \in \mathbb{Z}_p^\times$ and henceforth take $a \in \{1, \beta, p, p\beta\}$. Let η be either the trivial character or $\left(\frac{\cdot}{p}\right)$, assume the compatibility condition $\eta(-1) = \mu_a(-1)$, and put $\mu = \mu_a|\cdot|^{1/2}$. We write $c(\mu_a)$ and $c(\eta)$ for the respective conductor exponents.

Proposition 4.6 gives

$$0 \longrightarrow St_{\psi,\mu_a} \longrightarrow \pi_\psi(\mu) \xrightarrow{\mathcal{M}} \omega^+_{\psi,\mu_a} \longrightarrow 0.$$

For every level at which η defines a character of $\overline{K_m}$, let $\mathcal{M}_m$ be the restriction of $\mathcal{M}$ to the $(\overline{K_m}, \eta)$-isotypic space. By [16, Lemma 4.9], the map $\mathcal{M}_m$ is surjective for every m. We therefore have a short exact sequence

$$0 \longrightarrow (St_{\psi,\mu_a})^{K_m}_\eta \longrightarrow \pi_\psi(\mu)^{K_m}_\eta \xrightarrow{\mathcal{M}_m} (\omega^+_{\psi,\mu_a})^{K_m}_\eta \longrightarrow 0. \tag{4}$$

This is also an exact sequence of modules for the corresponding Hecke algebra H_m. Indeed, since $\mathcal{M}$ is a $\widetilde{G}$-intertwining map, for $T \in H_m$ and $v \in \pi_\psi(\mu)^{K_m}_\eta$ one has

$$\mathcal{M}_m(T \cdot v) = \mathcal{M}_m\left(\int_{\widetilde{G}} T(g)\pi_\psi(\mu)(g)v\, dg\right) = T \cdot \mathcal{M}_m(v).$$

Thus both the Steinberg kernel and the even Weil quotient are preserved by the Hecke action.

The compatibility condition leaves the following four cases:

(1) $c(\mu_a) = c(\eta) = 0$: $a \in \{1, \beta\}$ and $\eta = 1$;
(2) $c(\mu_a) = 1$, $c(\eta) = 0$: $a \in \{p, p\beta\}$, $\eta = 1$, and necessarily $p \equiv 1 \pmod 4$;
(3) $c(\mu_a) = 0$, $c(\eta) = 1$: $a \in \{1, \beta\}$, $\eta = \left(\frac{\cdot}{p}\right)$, and necessarily $p \equiv 1 \pmod 4$;
(4) $c(\mu_a) = c(\eta) = 1$: $a \in \{p, p\beta\}$ and $\eta = \left(\frac{\cdot}{p}\right) = \mu_a|_{\mathbb{Z}_p^\times}$.

The dimension formulas for the induced and even Weil representations are [16, Theorems 4.1 and 4.4]; in particular,

$$c_\eta(\omega^+_{\psi,\mu_a}) = 2c(\eta\mu_a) + c(\mu_a).$$

Together with (4), these formulas give the following conductor and dimension data.

Proposition 4.7. *In Cases* 1–4*, respectively, the η-conductors are*

Case	$c_\eta(\pi_\psi(\mu))$	$c_\eta(\omega^+_{\psi,\mu_a})$	$c_\eta(St_{\psi,\mu_a})$
1	0	0	1
2	2	3	2
3	2	2	2
4	1	1	1

The dimensions at the levels needed below are

Case	m	$\dim \pi_\psi(\mu)^{K_m}_\eta$	$\dim(\omega^+_{\psi,\mu_a})^{K_m}_\eta$	$\dim(St_{\psi,\mu_a})^{K_m}_\eta$
1	0	1	1	0
1	1	2	1	1
2	2	2	0	2
3	2	2	1	1
4	1	2	1	1

In Case 2 all three isotypic spaces vanish for $m < 2$. In Case 3 the three $(\overline{K_1}, \eta)$-isotypic spaces vanish. For the ramified type in Cases 3 and 4, we only use levels $m \geq 1$, where η is a character of $\overline{K_m}$.

For each level under consideration, normalize Haar measure by $\mathrm{vol}(\overline{K_m}) = 1$. For $m \geq 2$, Section 3 shows that H_m is finite-dimensional, commutative, and semisimple; the level-1 Hecke subalgebras supported in $\overline{K}$ are studied in [2]. Thus the finite-dimensional $(\overline{K_m}, \eta)$-isotypic spaces admit simultaneous eigenbases for the action of H_m. We will explicitly distinguish the affine Iwahori operator U_1, whose support is not contained in $\overline{K}$, when it is used in Case 4.

With $\epsilon_p = \gamma(p, \psi)$ as fixed in Section 2, the conductor 0 normalization satisfies $\epsilon_p^2 = \left(\frac{-1}{p}\right)$.

Proposition 4.8. *Let the notation be as above. The following Hecke actions hold on the minimal-level Steinberg spaces:*

1. ***Case 1** ($c(\mu_a) = 0, c(\eta) = 0$): At level K_1, the Iwahori-Hecke operator $\widetilde{Q}_p$ acts on the one-dimensional Steinberg component with eigenvalue -1.*
2. ***Case 2** ($c(\mu_a) = 1, c(\eta) = 0$): At level K_2, the two-dimensional Steinberg component has a common eigenbasis on which $\mathcal{Y}_1$ acts by 0 and $\mathcal{W}_1$ acts by $\sqrt{p^*}$ and $-\sqrt{p^*}$, respectively.*
3. ***Case 3** ($c(\mu_a) = 0, c(\eta) = 1$): At level K_2, the operator $\mathcal{Y}_1$ acts by 0 on both the one-dimensional Steinberg component and the one-dimensional even Weil component. On the even Weil component $\mathcal{W}_1$ acts by*
$$\left(\frac{a}{p}\right)\sqrt{p^*},$$
and on the Steinberg component it acts by
$$-\left(\frac{a}{p}\right)\sqrt{p^*}.$$
4. ***Case 4** ($c(\mu_a) = 1, c(\eta) = 1$): At level K_1, the affine Iwahori operator U_1, supported on $K_1 w(p^{-1}) K_1$ and normalized by $U_1((w(p^{-1}), 1)) = \epsilon_p$, acts on the one-dimensional Steinberg component with eigenvalue $-\epsilon_p$.*

Proof. We analyze the action of the Hecke operators in each case by considering the natural projections between successive levels.

For **Case 1**, let f_1 and f_w be the standard basis of $\pi_\psi(\mu)_1^{K_1}$ supported on BK_1 and $B(w(1), 1)K_1$, normalized by $f_1(1) = f_w((w(1), 1)) = 1$. Let $\widetilde{Q}_p$ be the Iwahori Hecke operator supported on $K_1 w(1) K_1$ and normalized by $\widetilde{Q}_p((w(1), 1)) = 1$. The standard right-coset calculation gives
$$\widetilde{Q}_p f_1 = f_w, \qquad \widetilde{Q}_p f_w = p f_1 + (p-1) f_w.$$
Consequently
$$\widetilde{Q}_p(f_1 + f_w) = p(f_1 + f_w), \qquad \widetilde{Q}_p(p f_1 - f_w) = -(p f_1 - f_w).$$
The decomposition $K_0 = K_1 \sqcup K_1 w(1) K_1$ shows that $\widetilde{Q}_p + I$ is supported on $\overline{K_0}$ and takes the value ϵ at (k, ϵ) for $k \in K_0$. Its restriction to the

split copy $K_0 \times \{1\}$ is therefore the characteristic function of K_0. Since $\mathrm{vol}(\overline{K_1}) = 1$, we have $\mathrm{vol}(\overline{K_0}) = [K_0 : K_1] = p+1$. Consequently convolution by $(p+1)^{-1}(\widetilde{Q}_p + I)$ is the normalized averaging projection from the $(\overline{K_1}, 1)$-fixed space onto the $(\overline{K_0}, 1)$-fixed space. The two displayed identities show explicitly that this projector fixes the line $\mathbb{C}(f_1 + f_w)$ and annihilates the line $\mathbb{C}(pf_1 - f_w)$. The former is the $(\overline{K_0}, 1)$-fixed line $\pi_\psi(\mu)_1^{K_0}$. Since $\mathcal{M}_0$ is surjective and $(St_{\psi,\mu_a})_1^{K_0} = 0$, it maps isomorphically onto $(\omega^+_{\psi,\mu_a})_1^{K_0}$. Moreover, $(\omega^+_{\psi,\mu_a})_1^{K_0}$ and $(\omega^+_{\psi,\mu_a})_1^{K_1}$ are both one-dimensional, so they coincide. Thus the even Weil $(\overline{K_1}, 1)$-fixed vector is already an oldvector coming from level K_0. Exactness of (4) therefore identifies the latter line, $\mathbb{C}(pf_1 - f_w)$, with the Steinberg newvector line.

For **Case 2**, Proposition 4.7 gives $(\omega^+_{\psi,\mu_a})_1^{K_2} = 0$. Hence the entire two-dimensional induced space at level K_2 is the Steinberg space. Proposition 4.5, applied with $n = 1$, gives a common eigenbasis on which

$$\mathcal{Y}_1 = 0, \qquad \mathcal{W}_1 = \sqrt{p^*} \quad \text{and} \quad -\sqrt{p^*}.$$

For **Case 3**, Proposition 4.7 gives a two-dimensional induced space at level K_2, containing one Steinberg line and mapping onto one even Weil line, whereas all three $(\overline{K_1}, \eta)$-isotypic spaces vanish. By Remark 1, the element $\mathcal{Y}_1 = \mathcal{I} + \mathcal{V}_{1,1} + \mathcal{V}_{1,\lambda}$ is supported on $\overline{K_1}$ and takes the value $\eta(\tilde{k})^{-1}$ at each $\tilde{k} \in \overline{K_1}$. Under the normalization $\mathrm{vol}(\overline{K_2}) = 1$, one has $\mathrm{vol}(\overline{K_1}) = [K_1 : K_2] = p$, so $p^{-1}\mathcal{Y}_1$ is the normalized averaging projection onto the $(\overline{K_1}, \eta)$-isotypic space. That space is zero in Case 3; therefore

$$\mathcal{Y}_1 = 0$$

on the entire two-dimensional induced space.

We now adapt the argument used in Proposition 4.5; its hypotheses do not apply literally here because μ_a is unramified. Let f_1 and f_λ be the two standard type functions supported on $B(y(p), 1)K_2$ and $B(y(\lambda p), 1)K_2$, respectively. The terminal relation from Corollary 3.6 gives

$$\mathcal{W}_1^2 = p^*\mathcal{I}$$

on this space. Thus $\mathcal{W}_1$ is diagonalizable and its eigenvalues belong to $\{\sqrt{p^*}, -\sqrt{p^*}\}$.

It remains to show that both eigenvalues occur. The same direct double-coset calculation, now with $c(\mu_a) = 0$ and $c(\eta) = 1$, gives the equality of the two diagonal coefficients

$$\mathcal{V}_{1,1}(f_1)(\bar{y}(p)) = \mathcal{V}_{1,\lambda}(f_\lambda)(\bar{y}(\lambda p)). \tag{**}$$

Suppose that only one $\mathcal{W}_1$-eigenvalue occurred. Since $\mathcal{W}_1$ is diagonalizable, it would be scalar. Because $\mathcal{Y}_1 = 0$, the identities

$$\mathcal{V}_{1,1} + \mathcal{V}_{1,\lambda} = -\mathcal{I}, \qquad \mathcal{V}_{1,1} - \mathcal{V}_{1,\lambda} = \mathcal{W}_1$$

would then make both $\mathcal{V}_{1,1}$ and $\mathcal{V}_{1,\lambda}$ scalar. Equality $(**)$ would force their two scalar values to be equal, and hence $\mathcal{W}_1 = 0$, contrary to $\mathcal{W}_1^2 = p^*\mathcal{I}$. Thus the two eigenvalues of $\mathcal{W}_1$ are $\sqrt{p^*}$ and $-\sqrt{p^*}$.

We now determine which constituent receives which sign directly in the Schrödinger model. Write

$$\chi = \eta = \left(\frac{\cdot}{p}\right) \qquad \text{and} \qquad \phi_\eta(z) = \chi(z)\mathbf{1}_{\mathbb{Z}_p^\times}(z).$$

This character factor is essential: the plain characteristic function $\mathbf{1}_{\mathbb{Z}_p^\times}$ has trivial diagonal type. Since $a \in \{1, \beta\}$ is a unit, ψ_a has conductor 0 and $\gamma(u, \psi_a) = 1$ for every $u \in \mathbb{Z}_p^\times$. Hence

$$\omega_{\psi_a}((h(u), 1))\phi_\eta = \chi(u)\phi_\eta = \eta(h(u))\phi_\eta.$$

The verification of the three type conditions in the proof of [16, Theorem 4.4], applied here with $c(\psi_a) = 0$, $c(\eta) = 1$, and $m = 2$, shows that $x(\mathbb{Z}_p)$ fixes ϕ_η and that its Fourier transform is supported on $p^{-1}\mathbb{Z}_p^\times$; consequently $y(p^2\mathbb{Z}_p)$ also fixes ϕ_η. Thus ϕ_η belongs to the $(\overline{K_2}, \eta)$-isotypic space. Since $p \equiv 1 \pmod 4$, one has $\chi(-1) = 1$, and therefore ϕ_η is even. It consequently spans the one-dimensional space $(\omega_{\psi_a}^+)_\eta^{K_2}$.

We compute the action of $\mathcal{W}_1$ on this vector. For $\delta \in \{1, \lambda\}$, the right-coset decomposition in Lemma 2.4 gives

$$\begin{aligned}\mathcal{V}_{1,\delta}\phi_\eta &= \sum_{u \in \mathbb{Z}_p^\times/\sqrt{1+p\mathbb{Z}_p}} \eta(u)\omega_{\psi_a}((h(u), 1)(y(\delta p), 1))\phi_\eta \\ &= \sum_{u \in \mathbb{Z}_p^\times/\sqrt{1+p\mathbb{Z}_p}} \omega_{\psi_a}((y(\delta u^{-2}p), 1))\phi_\eta.\end{aligned}$$

Here we used $h(u)y(\delta p) = y(\delta u^{-2}p)h(u)$ and the two factors $\eta(u)$ and $\chi(u)$ multiply to 1, since $\eta = \chi$ is quadratic. As u^{-2} runs once through the nonzero squares in $\mathbb{F}_p$, subtraction of the two square classes gives

$$\mathcal{W}_1\phi_\eta = \sum_{t \in \mathbb{F}_p^*} \chi(t)\omega_{\psi_a}((y(pt), 1))\phi_\eta. \tag{5}$$

Put $\mathcal{F}_a = \omega_{\psi_a}((w(1), 1))$. Since the cover splits over K,

$$(y(pt), 1) = (w(1), 1)(x(-pt), 1)(w(1), 1)^{-1}$$

with no additional cocycle factor. Thus the right-hand side of (5) is

$$\mathcal{F}_a \left(\sum_{t \in \mathbb{F}_p^*} \chi(t)\omega_{\psi_a}((x(-pt), 1)) \right) \mathcal{F}_a^{-1}\phi_\eta.$$

Up to a nonzero constant,

$$[\mathcal{F}_a^{-1}\phi_\eta](z) = \int_{\mathbb{Z}_p^\times} \chi(x)\psi(-2axz)\, dx,$$

which is supported on $p^{-1}\mathbb{Z}_p^\times$. If $z = p^{-1}v$ with $v \in \mathbb{Z}_p^\times$, the operator in parentheses acts by the scalar

$$\begin{aligned}\sum_{t\in\mathbb{F}_p^*} \chi(t)\psi_a(-ptz^2) &= \sum_{t\in\mathbb{F}_p^*} \chi(t)\psi\left(-\frac{atv^2}{p}\right) \\ &= \chi(-a)\sum_{t\in\mathbb{F}_p^*} \chi(t)\psi(t/p) \\ &= \chi(a)\epsilon_p\sqrt{p} = \chi(a)\sqrt{p^*}.\end{aligned}$$

In the last line we used $\chi(-1) = 1$ and (2). It follows that

$$\mathcal{W}_1\phi_\eta = \left(\frac{a}{p}\right)\sqrt{p^*}\,\phi_\eta.$$

The exact sequence (4) is equivariant for the action of the Hecke subalgebra H_2. Since the Steinberg line and the even Weil quotient are the two opposite $\mathcal{W}_1$-eigenlines found above, $\mathcal{W}_1$ acts on the Steinberg component by $-\left(\frac{a}{p}\right)\sqrt{p^*}$.

Finally, in **Case 4** both the even Weil and Steinberg constituents have a one-dimensional $(\overline{K_1}, \eta)$-isotypic space. We use the affine Iwahori operator U_1 supported on $K_1 w(p^{-1})K_1$. This operator separates the two constituents, although it is not an element of the Hecke subalgebra supported on $\overline{K}$. We normalize

$$U_1\big((w(p^{-1}), 1)\big) = \epsilon_p,$$

where $\epsilon_p = \gamma(p, \psi)$ as above. The relation of [2, Proposition 3.10(ii)] remains valid with this normalization:

$$U_1^2 = \epsilon_p(p-1)U_1 + \chi_p(-1)pI.$$

In this case, put $w = w(1)$. Let f_1 and f_w be the standard basis of $\pi_\psi(\mu)_\eta^{K_1}$ supported on BK_1 and $B(w,1)K_1$, respectively, with the same normalization as in Case 1:

$$f_1(1) = 1, \qquad f_w((w,1)) = 1.$$

We have

$$\mu_a(-p) = \begin{cases} 1, & a = p, \\ -1, & a = p\beta. \end{cases}$$

Since

$$\mu_a(-1) = (-1, a)_p = (-1, p)_p = \left(\frac{-1}{p}\right),$$

we have

$$\mu_a(p) = \mu_a(-p)\left(\frac{-1}{p}\right).$$

Consequently,

$$[U_1(f)](g) = \epsilon_p \sum_{s\in\mathbb{F}_p} f\big(g(y(ps), 1)(w(p^{-1}), 1)\big).$$

We first compute $U_1(f_1)$ at $g = 1$. The term $s = 0$ lies in the $B(w,1)\overline{K_1}$-cell and therefore does not contribute. For $s \in \mathbb{F}_p^\times$, put

$$k_s = \begin{pmatrix} s & 0 \\ p & s^{-1} \end{pmatrix} \in K_1,$$
$$b_s := y(ps)w(p^{-1})k_s = x(p^{-1}s^{-1}) \in B_0.$$

Since $y(ps)w(p^{-1}) = b_s k_s^{-1}$, the character and cocycle factors are

$$\mu_B((b_s, 1)) = c(b_s, k_s^{-1}) = 1,$$
$$\eta((k_s^{-1}, 1)) = c(y(ps), w(p^{-1})) = \chi_p(s).$$

Consequently,

$$[U_1(f_1)](1) = \epsilon_p \sum_{s \in \mathbb{F}_p^\times} \chi_p(s)^2 = \epsilon_p(p-1).$$

We next evaluate at $g = w$. Since

$$w\, y(ps)\, w(p^{-1}) = \begin{pmatrix} -p & s \\ 0 & -p^{-1} \end{pmatrix} = h(-p)x(-sp^{-1}) \in B_0,$$

all terms contribute to the f_w-coefficient. The cocycle calculation in this case gives

$$c(w(1), y(ps)w(p^{-1})) = \begin{cases} 1, \text{ if } s = 0 \\ (s,p)_p, \text{ if } s \neq 0 \end{cases}$$

which for $s \neq 0$ equals $c(y(ps), w(p^{-1}))$, so there is no remaining cocycle contribution. Thus every summand is $\mu_B((h(-p)x(-sp^{-1}), 1))$. Since $\mu = \mu_a|\cdot|^{1/2}$,

$$\mu(-p) = \mu_a(-p)p^{-1/2}.$$

Moreover, $\gamma(-1, \psi) = 1$, $\gamma(p, \psi) = \epsilon_p$, and hence

$$\gamma(-p, \psi) = \gamma(-1, \psi)\gamma(p, \psi)(-1, p)_p = \epsilon_p \left(\frac{-1}{p}\right).$$

Consequently,

$$\begin{aligned} [U_1(f_1)](w) &= \epsilon_p \sum_{s \in \mathbb{F}_p} |-p|\, \mu(-p)\gamma(-p, \psi)^{-1} \\ &= \epsilon_p p \cdot p^{-1} \cdot \mu_a(-p)p^{-1/2} \left(\frac{-1}{p}\right) \epsilon_p^{-1} \\ &= \mu_a(-p) \left(\frac{-1}{p}\right) p^{-1/2}. \end{aligned}$$

Therefore

$$U_1(f_1) = \epsilon_p(p-1)f_1 + \mu_a(-p)\left(\frac{-1}{p}\right) p^{-1/2} f_w.$$

We now compute $U_1(f_w)$. Since

$$w\, y(ps)\, w(p^{-1}) = \begin{pmatrix} -p & s \\ 0 & -p^{-1} \end{pmatrix} \in B_0,$$

we have

$$[U_1(f_w)](w) = 0.$$

At $g = 1$,

$$y(ps)w(p^{-1}) = \begin{pmatrix} 0 & p^{-1} \\ -p & s \end{pmatrix}.$$

We claim that this lies in BwK_1 only when $s \equiv 0 \pmod p$. Indeed, suppose that

$$y(ps)w(p^{-1}) = bwk, \qquad k = \begin{pmatrix} \alpha & \beta_0 \\ \gamma & \delta \end{pmatrix} \in K_1.$$

Writing $b = h(r)x(u)$, comparison of the second row gives

$$\alpha = rp, \qquad \beta_0 = -rs.$$

If $s \not\equiv 0 \pmod p$, then $s \in \mathbb{Z}_p^\times$. Since $\beta_0 \in \mathbb{Z}_p$, it follows that $r \in \mathbb{Z}_p$, and hence

$$\alpha = rp \in p\mathbb{Z}_p.$$

Since also $\gamma \in p\mathbb{Z}_p$, this implies

$$\alpha\delta - \beta_0\gamma \in p\mathbb{Z}_p,$$

contradicting $\det(k) = 1$. Thus only $s = 0$ contributes.

For $s = 0$, we use only the Bw-decomposition of the element $w(p^{-1})$. In the metaplectic cover,

$$(w(p^{-1}), 1) = \left(h(p^{-1}), \left(\frac{-1}{p}\right)\right)(w, 1),$$

because

$$c(h(p^{-1}), w) = (-1, p)_p = \left(\frac{-1}{p}\right).$$

Therefore

$$[U_1(f_w)](1) = \epsilon_p\, p\, \mu(p^{-1}) \left(\frac{-1}{p}\right) \gamma(p^{-1}, \psi)^{-1}.$$

Since $\mu = \mu_a |\cdot|^{1/2}$ and μ_a is quadratic,

$$\mu(p^{-1}) = \mu_a(p^{-1})|p^{-1}|^{1/2} = \mu_a(-p)\left(\frac{-1}{p}\right) p^{1/2}.$$

Moreover, p^{-1} and p have the same square class, so

$$\gamma(p^{-1}, \psi) = \gamma(p, \psi) = \epsilon_p.$$

It follows that

$$\begin{aligned} [U_1(f_w)](1) &= \epsilon_p\, p\, \mu_a(-p)\left(\frac{-1}{p}\right)^2 p^{1/2}\epsilon_p^{-1} \\ &= \mu_a(-p)p^{3/2}. \end{aligned}$$

Therefore

$$U_1(f_w) = \mu_a(-p)p^{3/2} f_1.$$

Thus, with the usual convention that the columns record the images of f_1 and f_w, the matrix of U_1 in the ordered basis (f_1, f_w) is

$$[U_1]_{(f_1, f_w)} = \begin{pmatrix} \epsilon_p(p-1) & \mu_a(-p)p^{3/2} \\ \mu_a(-p)\left(\frac{-1}{p}\right) p^{-1/2} & 0 \end{pmatrix}.$$

Its characteristic polynomial is

$$\begin{aligned}\det(XI - U_1) &= X^2 - \epsilon_p(p-1)X - \chi_p(-1)p \\ &= (X - \epsilon_p p)(X + \epsilon_p).\end{aligned}$$

For both $a = p$ and $a = p\beta$, the eigenspaces are therefore:

Eigenvalue	**Eigenvector**
$\epsilon_p p$	$p^{3/2} f_1 + \mu_a(-p)\epsilon_p f_w$
$-\epsilon_p$	$p^{1/2} f_1 - \mu_a(-p)\epsilon_p f_w$

We now compute the action on the even Weil representation. Write $a = pu$, where $u \in \{1, \beta\}$, and let $\phi = \mathbf{1}_{\mathbb{Z}_p}$ in the Schrödinger model of $\omega^+_{\psi_a}$. Since $a = p$ or $p\beta$, the character ψ_a has conductor -1. The standard Weil formulas show that ϕ spans the one-dimensional $(\overline{K_1}, \eta)$-isotypic space of $\omega^+_{\psi_a}$. By the normalization of U_1,

$$[U_1\phi](y) = \epsilon_p \sum_{s \in \mathbb{F}_p} \omega_{\psi_a}\big((y(ps), 1)(w(p^{-1}), 1)\big)(\phi)(y).$$

Using

$$(y(ps), 1)(w(p^{-1}), 1)\left(I, \left(\frac{-1}{p}\right)\right) = (h(p^{-1}), 1)(w, 1)(x(-sp^{-1}), 1),$$

together with the Schrödinger-model formulas, we obtain

$$\begin{aligned}[U_1\phi](y) = \epsilon_p &\left(\frac{-1}{p}\right) p^{1/2} \gamma(p^{-1}, \psi_a)^{-1}\gamma(\psi_a) \\ &\times \sum_{s \in \mathbb{F}_p} \int_{\mathbb{Z}_p} \psi_a\left(-\frac{sz^2}{p}\right)\psi_a\left(\frac{2zy}{p}\right) d_{\psi_a,2}z.\end{aligned}$$

Since $c(\psi_a) = -1$, the self-dual measure is $d_{\psi_a,2}z = p^{-1/2}\,dz$. For $z, s \in \mathbb{Z}_p$,

$$\psi_a\left(-\frac{sz^2}{p}\right) = \psi(-usz^2) = 1, \qquad \psi_a\left(\frac{2zy}{p}\right) = \psi(2uzy),$$

so the sum over s contributes a factor p. It follows that

$$\begin{aligned}[U_1\phi](y) &= \epsilon_p \left(\frac{-1}{p}\right) p\, \gamma(p^{-1}, \psi_a)^{-1}\gamma(\psi_a) \int_{\mathbb{Z}_p} \psi(2uzy)\, dz \\ &= \epsilon_p \left(\frac{-1}{p}\right) p\, \gamma(p^{-1}, \psi_a)^{-1}\gamma(\psi_a)\, \mathbf{1}_{\mathbb{Z}_p}(y).\end{aligned}$$

By the definition of the relative Weil factor, and since ψ_u has conductor 0,

$$\gamma(p^{-1}, \psi_a) = \frac{\gamma(\psi_u)}{\gamma(\psi_a)} = \gamma(\psi_a)^{-1}.$$

Moreover, $\gamma(-1, \psi) = 1$ and $(-1, a)_p = \chi_p(-1)$, so the Weil-factor identities give

$$\begin{aligned}\gamma(p^{-1}, \psi_a)^{-1}\gamma(\psi_a) &= \gamma(\psi_a)^2 \\ &= \gamma(-1, \psi_a)^{-1} = \chi_p(-1).\end{aligned}$$

We conclude that

$$[U_1\phi](y) = \epsilon_p \chi_p(-1)^2 p\, \mathbf{1}_{\mathbb{Z}_p}(y) = \epsilon_p p\, \phi(y).$$

Comparing this with the preceding eigenspace decomposition and using the U_1-equivariance of (4), we conclude that the Steinberg kernel is the $-\epsilon_p$-eigenline, spanned by

$$p^{1/2} f_1 - \mu_a(-p)\epsilon_p f_w.$$

□

4.3. **Supercuspidal Representations.** We now consider irreducible genuine supercuspidal representations of $\widetilde{G}$. For the classification and the corresponding linear theory we refer to [16, §5.1], [13, §3.3], and the original work of Manderscheid [11, 12]. Let $\alpha = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and consider the maximal compact subgroups of G:

$$K^0 = K = \mathrm{SL}_2(\mathbb{Z}_p),$$

$$K^1 = \alpha^{-1} K^0 \alpha = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p) : \begin{array}{c} a, d \in \mathbb{Z}_p, \quad b \in p^{-1}\mathbb{Z}_p, \\ c \in p\mathbb{Z}_p \end{array} \right\}.$$

We retain $K = K^0 = \mathrm{SL}_2(\mathbb{Z}_p)$ throughout. When treating the two maximal compact subgroups uniformly, we write $\mathcal{K} \in \{K^0, K^1\}$.

We use the isomorphism $\iota : \widetilde{G}_c \to \widetilde{G}_{\sigma_0}$ from Section 2 to compare the compact splittings. For $\delta \in \{0, 1\}$, let $s^\delta_{\sigma_0}$ be Ishimoto's splitting of K^δ in the σ_0-model; see [16, §3.1]. Pulling it back through ι^{-1} gives

$$s^\delta_c(k) = s_p(k) s^\delta_{\sigma_0}(k), \qquad k \in K^\delta.$$

On the standard generators below, s_p equals 1: the lower-left entry is zero for $x(b)$ and $h(a)$; for $y(c)$ the factor is $(c, 1)_p = 1$; and $w(1)h(p^\delta)$ has lower-right entry zero. Thus the two cocycle models give the same splitting values on these generators. Writing $s^\delta = s^\delta_c$, we have:

s^0	s^1
$s^0(x(b)) = 1,\ b \in \mathbb{Z}_p$	$s^1(x(b)) = 1,\ b \in p^{-1}\mathbb{Z}_p$
$s^0(h(a)) = 1,\ a \in \mathbb{Z}_p^\times$	$s^1(h(a)) = \gamma(a, \psi_-),\ a \in \mathbb{Z}_p^\times$
$s^0(y(c)) = 1,\ c \in \mathbb{Z}_p$	$s^1(y(c)) = 1,\ c \in p\mathbb{Z}_p$
$s^0\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = 1$	$s^1\left(\begin{pmatrix} 0 & p^{-1} \\ -p & 0 \end{pmatrix}\right) = 1$

where ψ_- is the additive character of conductor -1 fixed in Section 2. We shall use the Weil-factor identities recalled there. For $u \in \mathbb{Z}_p^\times$, the normalization $c(\psi_-) = -1$ gives

$$s^1(h(u)) = \gamma(u, \psi_-) = \left(\frac{u}{p}\right).$$

For $\mathcal{K} = K^\delta$, write $s = s^\delta$. The splitting identifies irreducible representations of $\mathcal{K}$ with genuine irreducible representations of $\widetilde{\mathcal{K}}$. Explicitly, a representation (ρ, W) of $\mathcal{K}$ gives the genuine representation $\widetilde{\rho}$ defined by

$$\widetilde{\rho}((k, \epsilon)) = \epsilon s(k) \rho(k), \qquad k \in \mathcal{K}, \quad \epsilon \in \{\pm 1\}.$$

This notation will also be used after replacing ρ by a conjugate representation or by a representation induced from the Iwahori subgroup.

Put

$$N_r = \{x(t) : t \in p^r \mathbb{Z}_p\}, \qquad \overline{N}_r = \{y(t) : t \in p^r \mathbb{Z}_p\},$$

and, inside K^δ, consider

$$J_\ell^\delta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K^\delta : \begin{array}{l} a, d \in 1 + p^\ell \mathbb{Z}_p, \\ b \in p^{\ell-\delta}\mathbb{Z}_p, \quad c \in p^{\ell+\delta}\mathbb{Z}_p \end{array} \right\}.$$

Following [16, §5.1] and Manderscheid [11, 12], we define the conductor $c(\rho)$ of a representation ρ of K^δ as the minimal m such that ρ is trivial on J_m^δ. We say that ρ is strongly cuspidal if

$$\mathrm{Hom}_{N_{c(\rho)-2\delta-1}}\big(\rho|_{N_{c(\rho)-2\delta-1}}, 1\big) = 0.$$

A strongly cuspidal representation ρ has *defect* 1 if

$$\mathrm{Hom}_{N_{c(\rho)-\delta-1}}\big(\rho|_{N_{c(\rho)-\delta-1}}, 1\big) \neq 0;$$

otherwise it has *defect* 0. We denote the defect by $d(\rho)$. If $d(\rho) = 1$, then necessarily $\delta = 1$.

We use Manderscheid's classification [11, 12] in the form stated in [16, Theorem 5.2]. The isomorphism ι identifies the corresponding genuine lifts and preserves compact induction, so the theorem applies in our c-model: for every irreducible strongly cuspidal representation ρ of $\mathcal{K} = K^\delta$,

$$\text{c-Ind}_{\widetilde{\mathcal{K}}}^{\widetilde{G}} \widetilde{\rho}$$

is irreducible and genuine supercuspidal; every irreducible genuine supercuspidal representation of $\widetilde{G}$ arises in this way; and inequivalent strongly cuspidal types induce inequivalent representations. Our aim is to find the newvectors in these compact inductions.

We first recall the linear data from which the maximal-compact types used below arise. Let

$$K_{\mathrm{GL}} = \mathrm{GL}_2(\mathbb{Z}_p), \qquad I_{\mathrm{GL}} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{\mathrm{GL}} : c \in p\mathbb{Z}_p \right\},$$

let Z be the center of $\mathrm{GL}_2(\mathbb{Q}_p)$, and let Z' be the subgroup generated by

$$\varpi_I = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}.$$

Thus ZK_{GL} and $Z'I_{\mathrm{GL}}$ are the two compact-mod-center subgroups used in the Kutzko–Sally construction. Following [13, Definition 3.3.1], the linear very cuspidal data occur in two cases:

$$\begin{array}{rl} \text{unramified:} & \tau \text{ is a representation of } ZK_{\mathrm{GL}}, \\ \text{ramified:} & \tau \text{ is a representation of } Z'I_{\mathrm{GL}}. \end{array}$$

In the corresponding case, level $\ell \geq 1$ means that τ is trivial on

$$K_{\mathrm{GL}}(\ell) = 1 + p^\ell M_2(\mathbb{Z}_p)$$

or on

$$I_{\mathrm{GL}}(\ell) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I_{\mathrm{GL}} : a, d \in 1 + p^\ell\mathbb{Z}_p, \quad b \in p^\ell\mathbb{Z}_p, \quad c \in p^{\ell+1}\mathbb{Z}_p \right\},$$

and in either case

$$\mathrm{Hom}_{N_{\ell-1}}(1, \tau) = 0.$$

The restriction of an unramified datum τ to $K_{\mathrm{GL}} \cap G = K^0$ is usually irreducible; in the exceptional case $\ell = 1$ it may split into two irreducible constituents. The restriction of a ramified datum τ to

$$I = I_{\mathrm{GL}} \cap G$$

always splits as $\sigma_1 \oplus \sigma_2$, and conjugation by a diagonal nonsquare-unit element interchanges the two constituents; see [13, §3.3].

For clarity, we now define the level of these Iwahori constituents. Put

$$I_\ell = I_{\mathrm{GL}}(\ell) \cap G.$$

We say that a representation σ of I has *Iwahori level* ℓ if

$$\sigma|_{I_\ell} = 1 \qquad \text{and} \qquad \mathrm{Hom}_{N_{\ell-1}}(1, \sigma) = 0.$$

Each constituent of a ramified very cuspidal datum of level ℓ has Iwahori level ℓ. In addition, Lansky–Raghuram [13, Remark 3.3.2] prove for the ramified datum τ that

$$\mathrm{Hom}_{N_{\ell-1}}(1, \tau) = \mathrm{Hom}_{\overline{N}_\ell}(1, \tau) = 0.$$

Passing to either constituent gives

$$\mathrm{Hom}_{\overline{N}_\ell}(1, \sigma_i) = 0, \qquad i = 1, 2.$$

This lower-unipotent assertion uses the extension to the ramified very cuspidal representation of $Z' I_{\mathrm{GL}}$; it need not hold for an arbitrary representation of I having the two properties in the definition above.

We record one consequence needed for the odd ramified construction. If

$$\rho = \mathrm{Ind}_I^{K^0} \sigma_i,$$

then $J^0_{\ell+1} \subset I_\ell$ and the normality of $J^0_{\ell+1}$ in K^0 give $c(\rho) \le \ell + 1$. On the other hand, σ_i is trivial on $\overline{N}_{\ell+1} \subset I_\ell$ and has no $\overline{N}_\ell$-fixed vector. Hence ρ is nontrivial on J^0_ℓ, through

$$J^0_\ell / I_\ell \simeq \overline{N}_\ell / \overline{N}_{\ell+1},$$

and therefore

$$c(\rho) = \ell + 1.$$

The following observation also determines the defect after passing from K^0 to K^1.

Lemma 4.9. *Let σ be an irreducible Iwahori constituent of a ramified very cuspidal datum of level ℓ, and put*

$$\rho = \mathrm{Ind}_I^{K^0} \sigma, \qquad \rho^\alpha(k) = \rho(\alpha k \alpha^{-1}), \qquad k \in K^1.$$

Then ρ^α is an irreducible strongly cuspidal representation of K^1 with

$$c(\rho^\alpha) = \ell + 1, \qquad d(\rho^\alpha) = 1.$$

Proof. Put $I' = \alpha^{-1} I \alpha$ and define

$$\sigma^\alpha(i') = \sigma(\alpha i' \alpha^{-1}), \qquad i' \in I'.$$

Functoriality of induction under the isomorphism $\mathrm{Ad}_\alpha : K^1 \to K^0$ gives

$$\rho^\alpha \simeq \mathrm{Ind}_{I'}^{K^1} \sigma^\alpha.$$

Let

$$w_1 = w(p^{-1}) = \begin{pmatrix} 0 & p^{-1} \\ -p & 0 \end{pmatrix} \in K^1.$$

A direct matrix calculation gives

$$w_1 I' w_1^{-1} = I.$$

Conjugating the induced model by w_1 therefore gives

$$\rho^\alpha \simeq \mathrm{Ind}_I^{K^1} \sigma',$$

where

$$\begin{aligned} \sigma'(i) &= \sigma^\alpha(w_1^{-1} i w_1) \\ &= \sigma(\gamma i \gamma^{-1}), \qquad \gamma = \alpha w_1^{-1} = \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}. \end{aligned}$$

Since

$$\gamma = \varpi_I \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in Z' I_{\mathrm{GL}},$$

the representation σ' is again an irreducible constituent of the restriction of the same ramified very cuspidal datum to I. Ishimoto's [16, Lemma 5.5(2)], which records the corresponding Kutzko–Sally–Manderscheid result, now shows that $\mathrm{Ind}_I^{K^1} \sigma'$ is irreducible and strongly cuspidal, of conductor $\ell + 1$ and defect 1. □

We next introduce the weight spaces used in the explicit vectors. Put

$$U = \{h(u) : u \in \mathbb{Z}_p^\times\}.$$

For a character $\eta : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ satisfying the usual central compatibility condition, let

$$W_\eta^U = \{w \in W : \rho(h(u))w = \eta(u)w \text{ for every } u \in \mathbb{Z}_p^\times\}.$$

When a level ℓ is fixed below, we write η_ℓ for the character occurring in the linear U-weight space. In the unramified construction,

$$\eta_\ell = \begin{cases} \eta, & \ell \text{ even}, \\ \eta(\frac{\cdot}{p}), & \ell \text{ odd}, \end{cases}$$

while in the ramified normalization we use $\eta_\ell = \eta(\frac{\cdot}{p})$. Here η itself always denotes the character defining the $(\overline{K_r}, \eta)$-isotypic space and the corresponding Hecke action. We recall the dimensions of the spaces that occur in the supercuspidal constructions. By [13, Propositions 3.3.4, 3.3.6, 3.3.8] and the Kutzko–Sally classification recalled in [16, §5.1] (see also [10]), if the corresponding unramified supercuspidal L-packet has cardinality two, the restriction of the very cuspidal GL_2-type to K^0 is irreducible and $\dim W_{\eta_\ell}^U = 2$. For the exceptional unramified packet of cardinality four one has $\ell = 1$ and the restriction splits as $\sigma_1 \oplus \sigma_2$; for the space W_i of each irreducible constituent, $\dim(W_i)_{\eta_\ell}^U = 1$. In the ramified case the restriction to the Iwahori subgroup of SL_2 splits as $\sigma_1 \oplus \sigma_2$, and again $\dim(W_i)_{\eta_\ell}^U = 1$ for each constituent. Thus the two-dimensional U-weight space belongs to a single inducing type only in the unramified packet of cardinality two. Fix a nonsquare unit $\lambda \in \mathbb{Z}_p^\times$.

We shall keep three levels distinct. The integer ℓ is the level of the underlying very cuspidal or Iwahori datum. The associated strongly cuspidal maximal-compact type (ρ, W) has conductor and defect

$$(c(\rho), d(\rho)) = \begin{cases} (\ell, 0), & \text{in the unramified case,} \\ (\ell+1, 1), & \text{in the ramified case.} \end{cases}$$

Finally, for the choices of maximal compact subgroup made below and for $c(\eta) \leq 1$, the compact induction $\widetilde{\pi}$ has η-conductor 2ℓ in the unramified case and $2\ell + 1$ in the ramified case. For the unramified construction and the even ramified construction, the assertions about $(c(\rho), d(\rho))$ follow from [16, Lemma 5.5], and the η-conductors then follow from [16, Theorem 5.11 and Corollary 5.12]. The odd ramified construction in Proposition 4.13(iv) is our separate construction. Its conductor and defect are established in Lemma 4.9. Thus "level ℓ" for the linear datum, $c(\rho)$ for the maximal-compact type, and $c_\eta(\widetilde{\pi})$ are not interchangeable.

Lemma 4.10. *Let $\mathcal{K}$ be either K^0 or K^1, let (ρ, W) be a representation of $\mathcal{K}$, and let $\chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ be a character. Suppose that $w \in W_\chi^U$ is fixed by $\overline{N}_\ell$ and that*

$$\mathrm{Hom}_{\overline{N}_{\ell-1}}(1, \rho) = 0.$$

For $\delta \in \{1, \lambda\}$ define

$$\mathcal{T}_{\ell,\delta}(w) = \sum_{u \in \mathbb{Z}_p^\times / \sqrt{1+p\mathbb{Z}_p}} \chi(u)^{-1} \rho\big(h(u) y(\delta p^{\ell-1})\big) w.$$

Then $\mathcal{T}_{\ell,\delta}(w) \in W_\chi^U$ for each $\delta \in \{1, \lambda\}$, and

$$\big(I + \mathcal{T}_{\ell,1} + \mathcal{T}_{\ell,\lambda}\big) w = 0.$$

Proof. We give the calculation exactly in the linear model. For $\delta \in \{1, \lambda\}$ we have

$$\begin{aligned} \mathcal{T}_{\ell,\delta}(w) &= \sum_{u \in \mathbb{Z}_p^\times / \sqrt{1+p\mathbb{Z}_p}} \chi(u)^{-1} \rho\big(h(u) y(\delta p^{\ell-1})\big) w \\ &= \sum_{u \in \mathbb{Z}_p^\times / \sqrt{1+p\mathbb{Z}_p}} \chi(u)^{-1} \rho\big(y(\delta p^{\ell-1} u^{-2}) h(u)\big) w \\ &= \sum_{u \in \mathbb{Z}_p^\times / \sqrt{1+p\mathbb{Z}_p}} \rho\big(y(\delta p^{\ell-1} u^{-2})\big) w, \end{aligned}$$

where we used

$$h(u) y(t) = y(tu^{-2}) h(u)$$

and $\rho(h(u))w = \chi(u)w$.

Now

$$\overline{N}_{\ell-1}/\overline{N}_\ell \simeq p^{\ell-1}\mathbb{Z}_p / p^\ell \mathbb{Z}_p \simeq \mathbb{F}_p.$$

The zero class gives the identity operator. The nonzero classes split into the two square classes: the elements

$$\delta u^{-2} \pmod{p}, \qquad u \in \mathbb{Z}_p^\times / \sqrt{1 + p\mathbb{Z}_p}, \qquad \delta \in \{1, \lambda\},$$

run exactly once through $\mathbb{F}_p^*$. Hence, with Haar measure on $\overline{N}_{\ell-1}$ normalized by $\mathrm{vol}(\overline{N}_\ell) = 1$, the preceding two sums together with the identity term give

$$\big(I + \mathcal{T}_{\ell,1} + \mathcal{T}_{\ell,\lambda}\big) w = \int_{\overline{N}_{\ell-1}} \rho(n) w \, dn.$$

The last integral is a nonzero scalar multiple of the averaging projection onto the trivial $\overline{N}_{\ell-1}$-isotypic subspace. Since

$$\mathrm{Hom}_{\overline{N}_{\ell-1}}(1, \rho) = 0$$

by hypothesis, the integral is zero. Therefore

$$\big(I + \mathcal{T}_{\ell,1} + \mathcal{T}_{\ell,\lambda}\big) w = 0.$$

□

Corollary 4.11 (For the inducing types)**.** *(i) Let (ρ, W) be a strongly cuspidal representation of K^0 with $c(\rho) = \ell$ and $d(\rho) = 0$. Then every $w \in W_\chi^U$ is fixed by $\overline{N}_\ell$ and*

$$\mathrm{Hom}_{\overline{N}_{\ell-1}}(1, \rho) = 0.$$

Thus every such w satisfies the hypotheses of Lemma 4.10.

(ii) Let (ρ, W) be a strongly cuspidal representation of K^1 with $c(\rho) = \ell + 1$ and defect 1. Then

$$\mathrm{Hom}_{\overline{N}_\ell}(1, \rho) = 0.$$

Consequently, every $\phi \in W_\chi^U$ that is fixed by $\overline{N}_{\ell+1}$ satisfies Lemma 4.10 with $\ell + 1$ in place of ℓ.

Proof. For (i), the inclusion $\overline{N}_\ell \subset J_\ell^0$ shows that every vector of W is $\overline{N}_\ell$-fixed. Strong cuspidality gives $\mathrm{Hom}_{N_{\ell-1}}(1, \rho) = 0$, and conjugation by $w(1) \in K^0$ carries $N_{\ell-1}$ onto $\overline{N}_{\ell-1}$.

Here and below we use the equivalence between the absence of invariant vectors and the absence of invariant linear forms. It follows from complete reducibility for finite-dimensional representations of compact groups and reconciles this notation with the definition of strong cuspidality above.

For (ii), strong cuspidality gives $\mathrm{Hom}_{N_{\ell-2}}(1, \rho) = 0$. Conjugation by $w(p^{-1}) \in K^1$ carries $N_{\ell-2}$ onto $\overline{N}_\ell$, and hence $\mathrm{Hom}_{\overline{N}_\ell}(1, \rho) = 0$. Together with the assumed $\overline{N}_{\ell+1}$-fixedness of ϕ, these are precisely the two hypotheses of Lemma 4.10 with $\ell + 1$ in place of ℓ. □

Corollary 4.11 records the filtration facts needed for the types and vectors in Proposition 4.13; it will be used again in the Hecke-action calculation of Proposition 4.14.

We next give the uniform formula for the vectors in the compactly induced representation, exactly so that the cocycle verification is performed only once.

Lemma 4.12. *Let $\mathcal{K} = K^0$ or K^1, let $(\widetilde{\rho}, W)$ be a genuine representation of $\widetilde{\mathcal{K}}$, and put*

$$\widetilde{\pi} = \text{c-Ind}_{\widetilde{\mathcal{K}}}^{\widetilde{G}} \widetilde{\rho}.$$

For $w \in W$ define

$$f_w((g,\epsilon)) = \begin{cases} c(k,h(p^m))\,\epsilon\,\widetilde{\rho}((k,1))w, & g = kh(p^m), \quad k \in \mathcal{K}, \\ 0, & g \notin \mathcal{K}h(p^m). \end{cases}$$

Then $f_w \in \widetilde{\pi}$.

Proof. We have to prove the defining left $\widetilde{\mathcal{K}}$-equivariance. Namely, for $(k_1,\epsilon_1) \in \widetilde{\mathcal{K}}$ and $(g,\epsilon) \in \widetilde{G}$ we want

$$f_w\big((k_1,\epsilon_1)(g,\epsilon)\big) = \widetilde{\rho}((k_1,\epsilon_1))f_w((g,\epsilon)).$$

If $g \notin \mathcal{K}h(p^m)$, then also $k_1 g \notin \mathcal{K}h(p^m)$, and both sides are zero. Assume therefore that

$$g = kh(p^m), \qquad k \in \mathcal{K}.$$

Recall the cocycle identity

$$c(g_1,g_2)c(g_1g_2,g_3) = c(g_1,g_2g_3)c(g_2,g_3).$$

On the one hand,

$$(k_1,\epsilon_1)(g,\epsilon) = (k_1kh(p^m), \epsilon_1\epsilon c(k_1,g)),$$

so, by the definition of f_w,

$$f_w\big((k_1,\epsilon_1)(g,\epsilon)\big) = \epsilon\epsilon_1 c(k_1,g)c(k_1k,h(p^m))\widetilde{\rho}((k_1k,1))w.$$

On the other hand,

$$\begin{aligned} \widetilde{\rho}((k_1,\epsilon_1))f_w((g,\epsilon)) &= \epsilon\epsilon_1 c(k,h(p^m))\widetilde{\rho}((k_1,1))\widetilde{\rho}((k,1))w \\ &= \epsilon\epsilon_1 c(k,h(p^m))c(k_1,k)\widetilde{\rho}((k_1k,1))w. \end{aligned}$$

Thus it remains to prove

$$c(k_1,g)c(k_1k,h(p^m)) = c(k,h(p^m))c(k_1,k).$$

Since $g = kh(p^m)$, this is

$$c(k_1,kh(p^m))c(k_1k,h(p^m)) = c(k,h(p^m))c(k_1,k).$$

Taking $g_1 = k_1$, $g_2 = k$, $g_3 = h(p^m)$ in the cocycle identity gives exactly

$$c(k_1,k)c(k_1k,h(p^m)) = c(k_1,kh(p^m))c(k,h(p^m)).$$

Because all cocycle values are in $\{\pm 1\}$, this is equivalent to the required equality. Hence f_w satisfies the compact-induction transformation law and therefore belongs to $\widetilde{\pi}$. $\square$

In the following proposition, the linear representations under consideration are obtained from an unramified or ramified very cuspidal datum τ, as described above. In the ramified cases, ϕ_w is a vector in the indicated induced representation of K^0 or K^1. Case (iii) uses the passage from an Iwahori constituent to a strongly cuspidal K^1-type described in [16, Lemma 5.5(2)]. Case (iv) is a separate construction: we first induce from I to K^0 and then conjugate by α to obtain a representation of K^1.

Proposition 4.13. *Let*

$$\widetilde{\pi} = \text{c-Ind}_{\widetilde{\mathcal{K}}}^{\widetilde{G}}\,\widetilde{\rho},$$

where $\mathcal{K} = K^0$ in case (i) and $\mathcal{K} = K^1$ in cases (ii)–(iv). The vectors obtained from a strongly cuspidal type are described as follows.

(i) ***Unramified,*** ℓ ***even.*** *Write* $m = \ell/2$. *Let* (ρ, W) *be an irreducible strongly cuspidal representation of* K^0, *arising from an unramified datum of level* ℓ, *so that* $c(\rho) = \ell$ *and* $d(\rho) = 0$. *Let* $w \in W^U_{\eta_\ell}$, *and let* $\widetilde{\rho}$ *be the genuine lift of* ρ *with respect to* s^0. *Then*

$$f_w \in \widetilde{\pi}_\eta^{K_{2\ell}}.$$

(ii) ***Unramified,*** ℓ ***odd.*** *Write* $m = (\ell - 1)/2$. *Starting with an irreducible strongly cuspidal representation* (ρ, W) *of* K^0 *arising from an unramified datum of level* ℓ, *so that* $c(\rho) = \ell$ *and* $d(\rho) = 0$, *put*

$$\rho^\alpha(k) = \rho(\alpha k \alpha^{-1}), \qquad \alpha = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix},$$

view ρ^α *as a representation of* K^1, *and let* $\widetilde{\rho}$ *be its genuine lift with respect to* s^1. *The conjugated type again has conductor* ℓ *and defect* 0. *For the corresponding* $w \in W^U_{\eta_\ell}$ *one has*

$$f_w \in \widetilde{\pi}_\eta^{K_{2\ell}}.$$

(iii) ***Ramified,*** ℓ ***even.*** *Write* $m = \ell/2$. *Let* (σ, W) *be the Iwahori constituent arising from a ramified datum of level* ℓ *and put*

$$\rho = \mathrm{Ind}_I^{K^1} \sigma.$$

Then ρ *is irreducible and strongly cuspidal, with* $c(\rho) = \ell + 1$ *and* $d(\rho) = 1$, *by* [16, Lemma 5.5(2)]. *For* $w \in W^U_{\eta_\ell}$ *define* $\phi_w \in \rho$ *by*

$$\phi_w(k) = \begin{cases} \sigma(k)w, & k \in I, \\ 0, & k \notin I. \end{cases}$$

Then ϕ_w *is fixed by* N_ℓ *and* $\overline{N}_{\ell+1}$ *and* $\rho(h(u))\phi_w = \eta_\ell(u)\phi_w$. *Let* $\widetilde{\rho}$ *be the genuine lift of* ρ *with respect to* s^1. *Then*

$$f_{\phi_w} \in \widetilde{\pi}_\eta^{K_{2\ell+1}}.$$

(iv) ***Ramified,*** ℓ ***odd.*** *Write* $m = (\ell - 1)/2$. *Let* (σ, W) *be the corresponding conjugated Iwahori constituent arising from a ramified datum of level* ℓ *and put*

$$\rho = \mathrm{Ind}_I^{K^0} \sigma.$$

Take $w \in W^U_{\eta_\ell}$, *define* ϕ_w *as in (iii), and pass to the* α*-conjugate representation* ρ^α *on* K^1. *By Lemma 4.9,* ρ^α *is irreducible and strongly cuspidal, with*

$$c(\rho^\alpha) = \ell + 1, \qquad d(\rho^\alpha) = 1.$$

Let $\widetilde{\rho}$ *be the genuine lift of* ρ^α *with respect to* s^1. *Then*

$$f_{\phi_w} \in \widetilde{\pi}_\eta^{K_{2\ell+1}}.$$

Corollary 4.11 applies to every $w \in W^U_{\eta_\ell}$ *in the unramified cases. In the even ramified case, the displayed* $\overline{N}_{\ell+1}$*-fixedness of* ϕ_w*, together with part (ii) of that corollary, supplies the filtration hypotheses needed below. In the odd ramified case the assertion* $c(\rho^\alpha) = \ell + 1$, $d(\rho^\alpha) = 1$ *establishes the required maximal-compact type. The vector-level averaging needed for the later linear Hecke calculation follows directly from its* I*-supported induced model and is verified in Proposition 4.14. Since* $c(\eta) \leq 1$, *Theorem 5.11 of* [16] *gives*

$c_\eta(\widetilde{\pi}) = 2\ell$ in (i)–(ii) and $c_\eta(\widetilde{\pi}) = 2\ell+1$ in (iii)–(iv); Corollary 5.12 loc. cit. identifies the nonzero vectors constructed above as newvectors.

Proof. We treat the four cases separately. The right action on the compact induction is

$$[\widetilde{\pi}(\widetilde{g}_0)f](\widetilde{g}) = f(\widetilde{g}\widetilde{g}_0).$$

For $K_r = K_0(p^r)$ it is enough to examine the action of the upper unipotents $x(a)$ with $a \in \mathbb{Z}_p$, of the diagonal elements $h(u)$ with $u \in \mathbb{Z}_p^\times$, and of the lower unipotents $y(t)$ with $t \in p^r\mathbb{Z}_p$.

I. The unramified cases. Let $g = kh(p^m)$ with k in the relevant maximal compact subgroup.

(1) Upper unipotents. For $a \in \mathbb{Z}_p$ we have

$$h(p^m)x(a) = x(p^{2m}a)h(p^m),$$

and therefore

$$\begin{aligned}
[\widetilde{\pi}((x(a),1))f_w]((g,\epsilon)) &= f_w\big((g,\epsilon)(x(a),1)\big)\\
&= f_w\big((kx(p^{2m}a)h(p^m), \epsilon c(g,x(a)))\big)\\
&= c(kx(p^{2m}a), h(p^m))\, \epsilon c(g,x(a))\widetilde{\rho}((kx(p^{2m}a),1))w.
\end{aligned}$$

We now distinguish the parity of ℓ.

If ℓ is even, then $2m = \ell$ and $x(p^{2m}a) = x(p^\ell a) \in N_\ell$. Since w is N_ℓ-fixed,

$$\widetilde{\rho}((kx(p^{2m}a),1))w = \widetilde{\rho}((k,1))w.$$

For the chosen splitting over K^0, the cocycle factor $c(k, x(p^{2m}a))$ is 1, so this may also be written in the uniform form

$$\widetilde{\rho}((kx(p^{2m}a),1))w = c(k,x(p^{2m}a))\widetilde{\rho}((k,1))w.$$

If ℓ is odd, then $2m = \ell - 1$. Put $x' = x(p^{\ell-1}a)$. Since

$$\alpha x'\alpha^{-1} = x(p^\ell a) \in N_\ell,$$

we have $\rho^\alpha(x')w = w$. Moreover $s^1(x') = 1$, and the splitting identity gives

$$s^1(kx') = s^1(k)s^1(x')c(k,x') = s^1(k)c(k,x').$$

Consequently

$$\begin{aligned}
\widetilde{\rho}((kx',1))w &= s^1(kx')\rho^\alpha(kx')w\\
&= s^1(k)c(k,x')\rho^\alpha(k)w\\
&= c(k,x')\widetilde{\rho}((k,1))w.
\end{aligned}$$

Thus in both parity cases

$$\widetilde{\rho}((kx(p^{2m}a),1))w = c(k,x(p^{2m}a))\widetilde{\rho}((k,1))w.$$

It remains to prove

$$c(kx(p^{2m}a), h(p^m))c(kh(p^m), x(a)) = c(k,h(p^m))c(k,x(p^{2m}a)).$$

Set $x' = x(p^{2m}a)$. Since $x'h(p^m) = h(p^m)x(a)$, the cocycle identity gives

$$c(k,x')c(kx',h(p^m)) = c(k,x'h(p^m))c(x',h(p^m))$$

and

$$c(k,h(p^m))c(kh(p^m),x(a)) = c(k,h(p^m)x(a))c(h(p^m),x(a)).$$

The middle arguments are equal because $x'h(p^m) = h(p^m)x(a)$. A direct evaluation of the cocycle gives

$$c(x', h(p^m)) = c(h(p^m), x(a)) = 1.$$

Hence the required equality follows, and therefore

$$\widetilde{\pi}((x(a), 1))f_w = f_w.$$

(2) Diagonal units. Let $u \in \mathbb{Z}_p^\times$. Since $h(p^m)$ commutes with $h(u)$,

$$\begin{aligned}[\widetilde{\pi}((h(u), 1))f_w]((g, \epsilon)) &= f_w\big((kh(u)h(p^m), \epsilon c(g, h(u)))\big)\\ &= c(kh(u), h(p^m))\, \epsilon c(g, h(u))\widetilde{\rho}((kh(u), 1))w.\end{aligned}$$

For ℓ even, the U-weight condition gives directly

$$\widetilde{\rho}((kh(u), 1))w = \eta(u)c(k, h(u))\widetilde{\rho}((k, 1))w.$$

For ℓ odd we use the K^1-splitting and $w \in W^U_{\eta(\frac{\cdot}{p})}$:

$$\begin{aligned}\widetilde{\rho}((kh(u), 1))w &= s^1(kh(u))\rho^\alpha(kh(u))w\\ &= s^1(k)\left(\frac{u}{p}\right) c(k, h(u))\rho^\alpha(k)\eta(u)\left(\frac{u}{p}\right) w\\ &= \eta(u)c(k, h(u))\widetilde{\rho}((k, 1))w.\end{aligned}$$

Here we used $s^1(h(u)) = (\frac{u}{p})$ and $(\frac{u}{p})^2 = 1$.

We now compare the cocycles. From the cocycle identity,

$$c(k, h(u))c(kh(u), h(p^m)) = c(k, h(u)h(p^m))c(h(u), h(p^m))$$

and

$$c(k, h(p^m))c(kh(p^m), h(u)) = c(k, h(p^m)h(u))c(h(p^m), h(u)).$$

Since $h(u)h(p^m) = h(p^m)h(u)$, the first factors on the right are the same, and a direct calculation gives

$$c(h(u), h(p^m)) = c(h(p^m), h(u)).$$

It follows that

$$c(kh(u), h(p^m))c(kh(p^m), h(u)) = c(k, h(u))c(k, h(p^m)).$$

Substituting this in the preceding expression and using $c(k, h(u))^2 = 1$ yields

$$\widetilde{\pi}((h(u), 1))f_w = \eta(u)f_w.$$

(3) Lower unipotents. Let $t \in p^{2\ell}\mathbb{Z}_p$. Since

$$h(p^m)y(t) = y(p^{-2m}t)h(p^m),$$

we obtain

$$\begin{aligned}[\widetilde{\pi}((y(t), 1))f_w]((g, \epsilon)) &= f_w\big((ky(p^{-2m}t)h(p^m), \epsilon c(g, y(t)))\big)\\ &= c(ky(p^{-2m}t), h(p^m))\, \epsilon c(g, y(t))\widetilde{\rho}((ky(p^{-2m}t), 1))w.\end{aligned}$$

If ℓ is even, then $2m = \ell$ and

$$y(p^{-2m}t) \in \overline{N}_\ell,$$

so it fixes w. If ℓ is odd, then $2m = \ell - 1$ and

$$y(p^{-2m}t) \in \overline{N}_{\ell+1}.$$

Moreover

$$\alpha y(p^{-2m}t)\alpha^{-1} = y(p^{-2m-1}t) \in \overline{N}_\ell,$$

so it fixes w in the conjugated representation, and $s^1(y(p^{-2m}t)) = 1$. Therefore in both cases

$$\widetilde{\rho}((ky(p^{-2m}t), 1))w = c(k, y(p^{-2m}t))\widetilde{\rho}((k, 1))w.$$

Exactly as above, the cocycle identity reduces the desired equality to

$$c(y(p^{-2m}t), h(p^m))c(h(p^m), y(t)) = 1.$$

We check this explicitly. If $t = 0$, both cocycle values are 1. For $a \in \mathbb{Q}_p^\times$ and $x \in \mathbb{Q}_p^\times$, we have

$$c\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, y(x)\right) = c\left(y(x), \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = \begin{cases} (a, a^{-1})_p, & 2 \nmid \operatorname{ord}_p(ax), \\ (a, x)_p, & 2 \mid \operatorname{ord}_p(ax). \end{cases}$$

For $t \neq 0$, apply this with $a = p^m$ and with $x = p^{-2m}t$ and $x = t$, respectively. The two values of ax are $p^{-m}t$ and $p^m t$, whose valuations differ by $2m$ and hence have the same parity. If $2 \nmid \operatorname{ord}_p(p^{-m}t)$, both cocycles equal $(p^m, p^{-m})_p$, so their product is 1. If $2 \mid \operatorname{ord}_p(p^{-m}t)$, their product is

$$(p^m, p^{-2m}t)_p(p^m, t)_p = (p^m, t)_p^2 = 1,$$

since p^{-2m} is a square. Thus

$$c(y(p^{-2m}t), h(p^m))c(h(p^m), y(t)) = 1,$$

and consequently

$$\widetilde{\pi}((y(t), 1))f_w = f_w.$$

This proves (i) and (ii).

II. The ramified cases. We first verify the invariance properties of ϕ_w used in the calculation. If $n = x(a) \in N_\ell$, then, for $k \in I$,

$$[\rho(n)\phi_w](k) = \phi_w(kn) = \sigma(k)\sigma(n)w = \sigma(k)w = \phi_w(k),$$

and if $k \notin I$, then also $kn \notin I$, so both sides are zero. Hence ϕ_w is N_ℓ-fixed. The same argument with lower unipotents shows that ϕ_w is $\overline{N}_{\ell+1}$-fixed. The diagonal action is

$$\rho(h(u))\phi_w = \eta_\ell(u)\phi_w,$$

where in the ramified normalization

$$\eta_\ell(u) = \eta(u)\left(\frac{u}{p}\right).$$

Again let $g = kh(p^m)$.

(1) Upper unipotents. For $a \in \mathbb{Z}_p$ the matrix identity is the same:

$$h(p^m)x(a) = x(p^{2m}a)h(p^m).$$

Thus

$$\begin{aligned}[\widetilde{\pi}((x(a), 1))f_{\phi_w}]((g, \epsilon)) = c(kx(p^{2m}a), h(p^m))\, \epsilon c(g, x(a)) \\ \cdot \widetilde{\rho}((kx(p^{2m}a), 1))\phi_w.\end{aligned}$$

If ℓ is even, then $2m = \ell$ and $x(p^{2m}a) \in N_\ell$, so $\rho(x(p^{2m}a))\phi_w = \phi_w$. Since $s^1(x(p^{2m}a)) = 1$,

$$\begin{aligned}\widetilde{\rho}((kx(p^{2m}a),1))\phi_w &= s^1(kx(p^{2m}a))\rho(kx(p^{2m}a))\phi_w \\ &= s^1(k)c(k,x(p^{2m}a))\rho(k)\phi_w \\ &= c(k,x(p^{2m}a))\widetilde{\rho}((k,1))\phi_w.\end{aligned}$$

If ℓ is odd, then $2m = \ell - 1$ and we use the conjugated representation. Since

$$\alpha x(p^{2m}a)\alpha^{-1} = x(p^\ell a) \in N_\ell$$

and ϕ_w is N_ℓ-fixed, the same calculation gives

$$\widetilde{\rho}((kx(p^{2m}a),1))\phi_w = c(k,x(p^{2m}a))\widetilde{\rho}((k,1))\phi_w.$$

The remaining cocycle computation is exactly the one already written in Part I(1), and therefore

$$\widetilde{\pi}((x(a),1))f_{\phi_w} = f_{\phi_w}.$$

(2) Diagonal units. For $u \in \mathbb{Z}_p^\times$,

$$\begin{aligned}\widetilde{\rho}((kh(u),1))\phi_w &= s^1(kh(u))\rho(kh(u))\phi_w \\ &= s^1(k)s^1(h(u))c(k,h(u))\rho(k)\eta_\ell(u)\phi_w \\ &= \eta(u)c(k,h(u))\widetilde{\rho}((k,1))\phi_w,\end{aligned}$$

since $s^1(h(u)) = (\frac{u}{p})$ and $\eta_\ell(u) = \eta(u)(\frac{u}{p})$. The cocycle calculation is the same as Part I(2), so the required diagonal transformation by $\eta(u)$ follows.

(3) Lower unipotents. Now let $t \in p^{2\ell+1}\mathbb{Z}_p$. Again

$$h(p^m)y(t) = y(p^{-2m}t)h(p^m).$$

If ℓ is even, then $2m = \ell$ and

$$y(p^{-2m}t) \in \overline{N}_{\ell+1},$$

which fixes ϕ_w. If ℓ is odd, then $2m = \ell - 1$ and

$$y(p^{-2m}t) \in \overline{N}_{\ell+2},$$

while

$$\alpha y(p^{-2m}t)\alpha^{-1} = y(p^{-2m-1}t) \in \overline{N}_{\ell+1},$$

which also fixes ϕ_w. Since the splitting is 1 on these lower unipotents, in both cases

$$\widetilde{\rho}((ky(p^{-2m}t),1))\phi_w = c(k,y(p^{-2m}t))\widetilde{\rho}((k,1))\phi_w.$$

The rest of the calculation is identical to Part I(3): the same two uses of the cocycle identity reduce the equality to

$$c(y(p^{-2m}t),h(p^m))c(h(p^m),y(t)) = 1,$$

and the same Hilbert-symbol calculation proves it. Hence

$$\widetilde{\pi}((y(t),1))f_{\phi_w} = f_{\phi_w}.$$

This proves (iii) and (iv), and hence the proposition. $\square$

We now transfer the action of the highest nontrivial Hecke operators at the chosen newvector level to the preceding linear lemma. Thus, in the unramified case we work in the Hecke algebra at level $K_{2\ell}$ and use $\mathcal{V}_{2\ell-1,\delta}$; in the ramified case we work at level $K_{2\ell+1}$ and use $\mathcal{V}_{2\ell,\delta}$.

Proposition 4.14 (Metaplectic-to-linear Hecke reduction)**.** *Let $\delta \in \{1, \lambda\}$. We use the notation $\mathcal{T}_{r,\delta}$ from Lemma 4.10, with the linear U-character specified in each case below.*

(i) In the unramified cases of Proposition 4.13,

$$\mathcal{V}_{2\ell-1,\delta} f_w = f_{\mathcal{T}_{\ell,\delta} w}.$$

Consequently

$$\mathcal{Y}_{2\ell-1} f_w = \big(I + \mathcal{V}_{2\ell-1,1} + \mathcal{V}_{2\ell-1,\lambda}\big) f_w = 0.$$

(ii) In both ramified cases,

$$\mathcal{V}_{2\ell,\delta} f_{\phi_w} = f_{\mathcal{T}_{\ell+1,\delta}\phi_w}.$$

Consequently

$$\mathcal{Y}_{2\ell} f_{\phi_w} = \big(I + \mathcal{V}_{2\ell,1} + \mathcal{V}_{2\ell,\lambda}\big) f_{\phi_w} = 0.$$

Proof. We compute the Hecke action at the distinguished point $g = h(p^m)$ and then use $\widetilde{\mathcal{K}}$-equivariance. The right-coset decomposition used in the Hecke algebra gives

$$[\mathcal{V}_{r,\delta} f]((g,1)) = \sum_{u \in \mathbb{Z}_p^\times/\sqrt{1+p\mathbb{Z}_p}} \eta(u) f\big((g,1)(h(u),1)(y(\delta p^r),1)\big),$$

because the left biequivariance factor is $\overline{\eta((h(u),1))} = \overline{\eta(u^{-1})} = \eta(u)$. Here we use that η is unitary (and, in the present paper, quadratic). Here $r = 2\ell - 1$ in the unramified case and $r = 2\ell$ in the ramified case.

I. Unramified case, ℓ even. Here $m = \ell/2$ and $r = 2\ell - 1$. Since

$$h(p^m)h(u)y(\delta p^{2\ell-1}) = h(u)y(\delta p^{\ell-1})h(p^m),$$

with $h(u)y(\delta p^{\ell-1}) \in K$, the only point to check is that the central sign obtained by multiplying the three lifts is exactly cancelled by the cocycle $c(h(u)y(\delta p^{\ell-1}), h(p^m))$ appearing in the definition of f_w. Equivalently we must prove

$$c\big(h(p^m), h(u)y(\delta p^{2\ell-1})\big)c\big(h(u)y(\delta p^{\ell-1}), h(p^m)\big) = 1.$$

Put $t = \delta p^{2\ell-1}$ and $t' = p^{-2m}t = \delta p^{\ell-1}$. Since $h(p^m)h(u) = h(u)h(p^m)$ and $h(p^m)y(t) = y(t')h(p^m)$, the cocycle identity gives

$$\begin{aligned} c(h(p^m), h(u)y(t)) &= \frac{c(h(p^m), h(u))c(h(p^m)h(u), y(t))}{c(h(u), y(t))} \\ &= \frac{c(h(u), h(p^m)y(t))c(h(p^m), y(t))}{c(h(u), y(t))}, \\ c(h(u)y(t'), h(p^m)) &= \frac{c(h(u), y(t')h(p^m))c(y(t'), h(p^m))}{c(h(u), y(t'))} \\ &= \frac{c(h(u), h(p^m)y(t))c(y(t'), h(p^m))}{c(h(u), y(t'))}. \end{aligned}$$

Here we used $c(h(p^m),h(u)) = c(h(u),h(p^m))$. The diagonal–lower-unipotent formula is

$$c\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}, y(x)\right) = c\left(y(x), \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\right) = \begin{cases} (a,a^{-1})_p, & 2 \nmid \mathrm{ord}_p(ax), \\ (a,x)_p, & 2 \mid \mathrm{ord}_p(ax). \end{cases}$$

Since $t/t' = p^{2m}$ is a square, it gives $c(h(u),y(t)) = c(h(u),y(t'))$ and $c(h(p^m),y(t)) = c(y(t'),h(p^m))$. Therefore

$$c(h(p^m),h(u)y(t))c(h(u)y(t'),h(p^m)) = 1.$$

Thus the metaplectic factor is 1, and therefore

$$\begin{aligned}
[\mathcal{V}_{2\ell-1,\delta} f_w]((h(p^m),1)) &= \sum_{u\in\mathbb{Z}_p^\times/\sqrt{1+p\mathbb{Z}_p}} \eta(u)\widetilde{\rho}((h(u)y(\delta p^{\ell-1}),1))w \\
&= \sum_{u\in\mathbb{Z}_p^\times/\sqrt{1+p\mathbb{Z}_p}} \eta(u)\rho(h(u)y(\delta p^{\ell-1}))w \\
&= \mathcal{T}_{\ell,\delta} w.
\end{aligned}$$

II. Unramified case, ℓ odd. Now $m = (\ell-1)/2$ and again $r = 2\ell-1$. We have

$$h(p^m)h(u)y(\delta p^{2\ell-1}) = h(u)y(\delta p^{\ell})h(p^m).$$

The same cocycle calculation as above, with $t = \delta p^{2\ell-1}$ and $t' = \delta p^{\ell}$, shows that the total central factor is 1. Since

$$\alpha h(u)y(\delta p^{\ell})\alpha^{-1} = h(u)y(\delta p^{\ell-1}),$$

we get

$$\begin{aligned}
[\mathcal{V}_{2\ell-1,\delta} f_w]((h(p^m),1)) &= \sum_{u\in\mathbb{Z}_p^\times/\sqrt{1+p\mathbb{Z}_p}} \eta(u)s^1(h(u)y(\delta p^{\ell})) \\
&\qquad \cdot \rho^{\alpha}(h(u)y(\delta p^{\ell}))w \\
&= \sum_{u\in\mathbb{Z}_p^\times/\sqrt{1+p\mathbb{Z}_p}} \eta(u)\left(\frac{u}{p}\right) \\
&\qquad \cdot \rho(h(u)y(\delta p^{\ell-1}))w \\
&= \mathcal{T}_{\ell,\delta} w.
\end{aligned}$$

Here the second equality uses exactly the splitting relation

$$s^1(h(u)y(t)) = s^1(h(u))s^1(y(t))c(h(u),y(t)),$$

with $s^1(y(t)) = 1$ and $s^1(h(u)) = (\frac{u}{p})$, together with $w \in W_{\eta_\ell}^U$.

III. Ramified case, ℓ even. Here $m = \ell/2$ and the highest operator in the level-$2\ell+1$ Hecke algebra is $\mathcal{V}_{2\ell,\delta}$. We have

$$h(p^m)h(u)y(\delta p^{2\ell}) = h(u)y(\delta p^{\ell})h(p^m).$$

The metaplectic sign is

$$c\big(h(p^m),h(u)y(\delta p^{2\ell})\big)c\big(h(u)y(\delta p^{\ell}),h(p^m)\big),$$

and the same cocycle calculation as in Case I, now with $t = \delta p^{2\ell}$ and $t' = \delta p^{\ell}$, shows that this product is 1. Therefore

$$\begin{aligned}[\mathcal{V}_{2\ell,\delta} f_{\phi_w}]((h(p^m),1)) &= \sum_{u \in \mathbb{Z}_p^\times/\sqrt{1+p\mathbb{Z}_p}} \eta(u)\widetilde{\rho}((h(u)y(\delta p^\ell),1))\phi_w \\ &= \sum_{u \in \mathbb{Z}_p^\times/\sqrt{1+p\mathbb{Z}_p}} \eta(u)\left(\frac{u}{p}\right)\rho(h(u)y(\delta p^\ell))\phi_w \\ &= \sum_{u \in \mathbb{Z}_p^\times/\sqrt{1+p\mathbb{Z}_p}} \eta_\ell(u)^{-1}\rho(h(u)y(\delta p^\ell))\phi_w \\ &= \mathcal{T}_{\ell+1,\delta}\phi_w.\end{aligned}$$

This is precisely the linear calculation of Lemma 4.10 with ℓ replaced by $\ell+1$.

IV. Ramified case, ℓ odd. Now $m = (\ell-1)/2$ and again $r = 2\ell$. Thus

$$h(p^m)h(u)y(\delta p^{2\ell}) = h(u)y(\delta p^{\ell+1})h(p^m).$$

The K^1-type is the α-conjugate one, and

$$\alpha h(u)y(\delta p^{\ell+1})\alpha^{-1} = h(u)y(\delta p^\ell).$$

The same cocycle calculation applies with $t = \delta p^{2\ell}$ and $t' = \delta p^{\ell+1}$; the extra s^1-factor is cancelled by the splitting relation, exactly as in the odd unramified calculation. Hence

$$\begin{aligned}[\mathcal{V}_{2\ell,\delta} f_{\phi_w}]((h(p^m),1)) &= \sum_{u \in \mathbb{Z}_p^\times/\sqrt{1+p\mathbb{Z}_p}} \eta(u)\left(\frac{u}{p}\right)\rho(h(u)y(\delta p^\ell))\phi_w \\ &= \sum_{u \in \mathbb{Z}_p^\times/\sqrt{1+p\mathbb{Z}_p}} \eta_\ell(u)^{-1}\rho(h(u)y(\delta p^\ell))\phi_w \\ &= \mathcal{T}_{\ell+1,\delta}\phi_w.\end{aligned}$$

For an arbitrary point $kh(p^m)$ in the support, both sides of each asserted identity are obtained from their value at $h(p^m)$ by the same $\widetilde{\mathcal{K}}$-equivariance, and both vanish off the corresponding support. Thus the equalities hold as functions.

Finally, Lemma 4.10 gives in the unramified case

$$\big(I + \mathcal{V}_{2\ell-1,1} + \mathcal{V}_{2\ell-1,\lambda}\big) f_w = f_{(I+\mathcal{T}_{\ell,1}+\mathcal{T}_{\ell,\lambda})w} = 0.$$

It remains to justify the vanishing in the odd ramified case. Here the entire induced K^0-representation need not have zero $\overline{N}_\ell$-fixed space, so such vanishing does not hold in general. Nevertheless, the averaging projection kills the particular vector ϕ_w. Indeed, ϕ_w is again $\overline{N}_{\ell+1}$-fixed, because $\overline{N}_{\ell+1} \subset I_\ell$ and σ is trivial on I_ℓ. Moreover, ϕ_w is supported on I, and for $k \in I$,

$$\int_{\overline{N}_\ell} [\rho(n)\phi_w](k)\,dn = \sigma(k)\int_{\overline{N}_\ell} \sigma(n)w\,dn = 0,$$

where the last equality follows from $\mathrm{Hom}_{\overline{N}_\ell}(1,\sigma) = 0$, as proved above using [13, Remark 3.3.2]; outside I the integral is also zero. Thus the same averaging calculation gives

$$\big(I+\mathcal{V}_{2\ell,1}+\mathcal{V}_{2\ell,\lambda}\big)f_{\phi_w}=f_{(I+\mathcal{T}_{\ell+1,1}+\mathcal{T}_{\ell+1,\lambda})\phi_w}=0.$$

This proves the remaining odd ramified case. □

The packet descriptions used in the next proposition are those of [16, §5.1]; the relevant minimal fixed-space dimensions are given in [13, Propositions 3.3.4, 3.3.6, 3.3.8].

Let

$$a_\lambda=\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}\in \mathrm{GL}_2(\mathbb{Z}_p).$$

Conjugation by a_λ normalizes the relevant determinant-one compact subgroups, fixes every $h(u)$, and sends

$$y(t)\longmapsto y(\lambda^{-1}t).$$

Hence it preserves each U-weight space $W^U_{\eta_\ell}$ and interchanges the two square classes in the linear operators of Lemma 4.10.

Consider the operator $A_\lambda=\tau(a_\lambda)$, where τ is the very cuspidal representation introduced above. Then on the relevant U-weight spaces

$$A_\lambda\mathcal{T}_{\ell,1}A_\lambda^{-1}=\mathcal{T}_{\ell,\lambda},\qquad A_\lambda\mathcal{T}_{\ell,\lambda}A_\lambda^{-1}=\mathcal{T}_{\ell,1}.$$

In the cases where the restriction of the very cuspidal type splits into two constituents, the same nonsquare-determinant conjugation exchanges those two constituents; see [13, §3.3]. In the ramified cases, put $\mathcal{K}=K^0$ or K^1, as appropriate. If $A_\lambda W_i=W_j$, the corresponding map between the induced representations is defined by

$$\widehat{A}_\lambda:\mathrm{Ind}_I^{\mathcal{K}}\sigma_i\longrightarrow \mathrm{Ind}_I^{\mathcal{K}}\sigma_j,\qquad (\widehat{A}_\lambda\Phi)(k)=A_\lambda\Phi(a_\lambda^{-1}ka_\lambda).$$

This is well defined because a_λ normalizes I and $\mathcal{K}$, and $A_\lambda\sigma_i(a_\lambda^{-1}ia_\lambda)=\sigma_j(i)A_\lambda$ for $i\in I$. The definition of ϕ_w therefore gives

$$\widehat{A}_\lambda\phi_w=\phi_{A_\lambda w}.$$

On the induced spaces, the conjugation identities above are understood with $\widehat{A}_\lambda$ in place of A_λ. This is also compatible with the subsequent α-conjugation in the odd ramified case, since a_λ commutes with α.

No lift of a_λ to the metaplectic group is needed below: Proposition 4.14 transfers the resulting linear identities to the metaplectic newvectors.

Proposition 4.15. *Let η_ℓ be as above.*

(i) *Suppose that the supercuspidal L-packet is unramified of cardinality two. Then $\dim W^U_{\eta_\ell}=2$. Choose common Hecke eigenvectors w_1,w_2 spanning this space. The operator $\mathcal{W}_{2\ell-1}$ has distinct eigenvalues on f_{w_1} and f_{w_2}, namely*

$$\sqrt{p^*}\quad\text{and}\quad -\sqrt{p^*}$$

in some order.

(ii) *Suppose that the supercuspidal L-packet is the exceptional unramified packet of cardinality four. Then $\ell=1$ and the relevant restriction splits as $\sigma_1\oplus\sigma_2$. If $0\neq w_i\in(W_i)^U_{\eta_\ell}$ and f_{w_i} denotes the corresponding newvector for the packet member attached to σ_i, then $\mathcal{W}_1$ acts on these two newvectors by opposite eigenvalues $\sqrt{p^*}$ and $-\sqrt{p^*}$.*

(iii) *Suppose that the supercuspidal L-packet is ramified. Write the restriction of the very cuspidal type to the Iwahori subgroup of* SL_2 *as* $\sigma_1 \oplus \sigma_2$*. If* $0 \neq w_i \in (W_i)^U_{\eta_\ell}$ *and* $f_{\phi_{w_i}}$ *is the corresponding newvector for the packet member attached to* σ_i*, then* $\mathcal{W}_{2\ell}$ *acts on* $f_{\phi_{w_1}}$ *and* $f_{\phi_{w_2}}$ *by opposite eigenvalues* $\sqrt{p^*}$ *and* $-\sqrt{p^*}$*.*

Proof. Put
$$D_\ell = \mathcal{T}_{\ell,1} - \mathcal{T}_{\ell,\lambda}.$$
In (i), Proposition 4.14 gives
$$\mathcal{W}_{2\ell-1} f_w = f_{D_\ell w}$$
for every $w \in W^U_{\eta_\ell}$. Suppose that $\mathcal{W}_{2\ell-1}$ had the same eigenvalue c on the two newvectors f_{w_1}, f_{w_2}. Since w_1, w_2 form a basis of the two-dimensional space $W^U_{\eta_\ell}$ and the map $w \mapsto f_w$ is injective, Proposition 4.14 implies
$$D_\ell = cI$$
on $W^U_{\eta_\ell}$. On the other hand, the linear conjugation above gives
$$A_\lambda D_\ell A_\lambda^{-1} = -D_\ell.$$
Because A_λ preserves $W^U_{\eta_\ell}$, conjugating the scalar identity $D_\ell = cI$ yields $cI = -cI$, and hence $c = 0$. Thus $D_\ell = 0$ on $W^U_{\eta_\ell}$. Applying Proposition 4.14 once more gives
$$\mathcal{W}_{2\ell-1} = 0$$
on the metaplectic newspace, contradicting the terminal relation
$$\mathcal{W}^2_{2\ell-1} = p^* I.$$
Therefore the two eigenvalues are distinct, and the same terminal relation shows that they are $\sqrt{p^*}$ and $-\sqrt{p^*}$.

For (ii), the two one-dimensional U-weight spaces lie in the two constituents exchanged by a_λ. The linear conjugation interchanges $\mathcal{T}_{1,1}$ and $\mathcal{T}_{1,\lambda}$, and therefore sends
$$D_1 = \mathcal{T}_{1,1} - \mathcal{T}_{1,\lambda}$$
to $-D_1$. Hence the D_1-eigenvalues on the two U-weight lines are opposite. Proposition 4.14 transfers these to opposite $\mathcal{W}_1$-eigenvalues on the corresponding metaplectic newvectors. Since $\mathcal{W}_1^2 = p^* I$, those eigenvalues are $\sqrt{p^*}$ and $-\sqrt{p^*}$.

The ramified case (iii) is identical, with
$$D_{\ell+1} = \mathcal{T}_{\ell+1,1} - \mathcal{T}_{\ell+1,\lambda}.$$
The nonsquare-determinant conjugation exchanges the two linear constituents and sends $D_{\ell+1}$ to $-D_{\ell+1}$. By Proposition 4.14, their eigenvalues transfer to opposite eigenvalues of $\mathcal{W}_{2\ell}$ on $f_{\phi_{w_1}}$ and $f_{\phi_{w_2}}$. Finally
$$\mathcal{W}^2_{2\ell} = p^* I$$
gives the two values $\sqrt{p^*}$ and $-\sqrt{p^*}$. □

5. Relation with Ueda's Twisting Operator

We finish by explaining how the operator $\mathcal{W}_{m-1}$ of this paper is related to the twisting operators in Ueda's classical theory and to the operator R_p in Ishimoto's *Kohnen–Ueda local newforms* [17]. This comparison is not used in the proofs above. It is included to clarify the expected global role of the operators from our Hecke subalgebra.

Throughout this section, let $m \geq 2$, and let η be the genuine extension to $\overline{K_m}$ of a character η_0 which is either trivial or $(\frac{\cdot}{p})$, as in Section 2. In particular, $c(\eta_0) \leq 1$. We retain the notation $V_\eta^{K_m}(\pi)$ for the fixed-type space defined in Section 2.

Ueda's operator twists Fourier coefficients, while Ishimoto's R_p is a local unipotent average. Under the present hypotheses, Ishimoto's operator preserves the $(\overline{K_m}, \eta)$-fixed space. Our operator $\mathcal{W}_{m-1}$ is defined by a Hecke function supported in $\overline{K}$. We relate the restriction of Ishimoto's operator to the action of $\mathcal{W}_{m-1}$, up to an explicit scalar, by a lifted GL_2-conjugation. The conjugated character is denoted by η^{J_m} below; it equals η when m is even and is the other quadratic type when m is odd.

5.1. **Ueda's operator and Ishimoto's local operator.** We recall Ueda's classical twisting operator in the special case needed here. If

$$f(z) = \sum_{n \geq 0} a(n) e^{2\pi i n z}$$

is a half-integral weight modular form and ϕ is a Dirichlet character, Ueda defines

$$f|R_\phi(z) = \sum_{n \geq 0} \phi(n) a(n) e^{2\pi i n z};$$

see [7, Section 0(c)] and [8]. For the quadratic character $\phi = (\frac{\cdot}{p})$, this gives

$$f|R_p(z) = \sum_{n \geq 0} \left(\frac{n}{p}\right) a(n) e^{2\pi i n z},$$

where $(\frac{n}{p}) = 0$ when $p \mid n$.

Equivalently, if

$$\tau_p = \sum_{a \in \mathbb{F}_p^*} \left(\frac{a}{p}\right) e^{2\pi i a/p}$$

is the usual quadratic Gauss sum, then

$$f|R_p(z) = \frac{1}{\tau_p} \sum_{a \in \mathbb{F}_p^*} \left(\frac{a}{p}\right) f\left(z + \frac{a}{p}\right).$$

Indeed, the coefficient of $e^{2\pi i n z}$ on the right is

$$\frac{1}{\tau_p} \sum_{a \in \mathbb{F}_p^*} \left(\frac{a}{p}\right) e^{2\pi i a n/p} a(n) = \left(\frac{n}{p}\right) a(n),$$

with value 0 when $p \mid n$.

For an irreducible genuine smooth representation (π_{σ_0}, V) of $\widetilde{G}_{\sigma_0}$, Ishimoto's operator R_ϖ is the corresponding local unipotent average. Let ι be the isomorphism between the two cocycle models defined in Section 2, and put $\pi = \pi_{\sigma_0} \circ \iota$. Since $s_p(x(t)) = 1$, we have $\iota((x(t), 1)) = (x(t), 1)$. Moreover, ι carries the canonical splitting of K_m in our model to the compact splitting used by Ishimoto. Thus the corresponding $(\overline{K_m}, \eta)$-type spaces agree, and Ishimoto's integral has the same formula in the c-model used in this paper. We take the standard normalization $\mathrm{vol}(\mathbb{Z}_p) = 1$. Then, in the notation of [17], after specializing to $F = \mathbb{Q}_p$ and $\varpi = p$, it is

$$R_p^\pi v = \int_{\mathbb{Z}_p^\times} \left(\frac{u}{p}\right) \pi((x(p^{-1}u), 1))v\, du;$$

we are using the notation R_p^π for R_ϖ to distinguish it from Ueda's classical operator R_p. In Ishimoto's formulation, this operator acts on the increasing union $\bigcup_{r \geq c(\eta_0)} V_\eta^{K_r}(\pi)$, but need not preserve each individual level. More precisely, [17, Proposition 3.7(b)] gives

$$R_p^\pi\big(V_\eta^{K_r}(\pi)\big) \subseteq V_\eta^{K_{r'}}(\pi), \qquad r' = \max\{2, c(\eta_0) + 1, r\}.$$

Under our assumptions $c(\eta_0) \leq 1$ and $m \geq 2$, taking $r = m$ gives $r' = m$. Thus R_p^π preserves $V_\eta^{K_m}(\pi)$, and the comparison below uses its restriction to this space.

For $u \in \mathbb{Z}_p^\times$, write $u = a + pb$ with $a \in \{1, \ldots, p-1\}$ and $b \in \mathbb{Z}_p$. Then $x(p^{-1}u) = x(a/p)x(b)$. Since $(x(b), 1)$ acts trivially on $V_\eta^{K_m}(\pi)$ and $(\frac{u}{p}) = (\frac{a}{p})$, the integrand is constant on each such residue class, which has measure p^{-1}. Hence

$$R_p^\pi v = \frac{1}{p} \sum_{a \in \mathbb{F}_p^*} \left(\frac{a}{p}\right) \pi((x(a/p), 1))v. \tag{6}$$

We give a direct adelic-to-classical comparison, following [2, §4]. Assume that f belongs to a classical space of weight $k + 1/2$ whose adelization Φ_f has right $(\overline{K_m}, \eta)$-type at p. Equivalently, the p-component of its adelic nebentypus induces the genuine character η on $\overline{K_m}$. Let ρ_p denote right translation at p, and define $R_p^{\rho_p}$ by the same local integral as R_p^π above. Choose $\widetilde{g}_\infty \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$ with $\widetilde{g}_\infty(i) = z$, and put $A_a = x(-a/p) \in \mathrm{SL}_2(\mathbb{Q})$ for $a = 1, \ldots, p-1$. Writing $s_\mathbb{Q}$ for the canonical rational embedding, left automorphy and (6) give

$$\begin{aligned}(R_p^{\rho_p}\Phi_f)(\widetilde{g}_\infty) &= \frac{1}{p}\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \Phi_f\big(\widetilde{g}_\infty(x(a/p), 1)\big) \\ &= \frac{1}{p}\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \Phi_f\big(s_\mathbb{Q}(A_a)\widetilde{g}_\infty(x(a/p), 1)\big).\end{aligned}$$

The three factors in the last expression are embedded diagonally, at the real place, and at the p-place, respectively. The p-component is the identity; at every other finite place the component is integral upper unipotent and acts trivially. The real component is $(A_a, 1)\widetilde{g}_\infty$, sending i to $z - a/p$. Since $(A_a, 1)$ has automorphy factor 1, the classical realization is

$$(R_p^{\rho_p}\Phi_f)(\widetilde{g}_\infty)J(\widetilde{g}_\infty, i)^{2k+1} = \frac{1}{p}\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) f\left(z - \frac{a}{p}\right) = g_p^-\,(f|R_p)(z),$$

where J is the half-integral-weight automorphy factor of [2, §4], and

$$\tau_p^- = \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) e^{-2\pi i a/p}, \qquad g_p^- := p^{-1}\tau_p^-.$$

Thus, after division by the scalar $g_p^- = p^{-1}\tau_p^-$, Ishimoto's local operator becomes Ueda's classical twisting operator. If one uses the opposite translation convention, $z - a/p$ is replaced by $z + a/p$ and τ_p^- by τ_p.

On Whittaker coefficients, the normalized integral multiplies the coefficient indexed by a p-adic unit by the Legendre symbol and kills the coefficients whose index is divisible by p.

5.2. **The lifted GL_2-conjugation.** Put

$$J_m = \begin{pmatrix} 0 & 1 \\ -p^m & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p).$$

A direct calculation gives

$$J_m K_m J_m^{-1} = K_m, \qquad J_m x(t) J_m^{-1} = y(-p^m t),$$

and

$$J_m y(t) J_m^{-1} = x(-p^{-m} t), \qquad J_m h(u) J_m^{-1} = h(u^{-1}).$$

We now explain the lift of this conjugation to the double cover used in this paper.

Using the notation $\tau(A)$ from Section 2 for $A \in \mathrm{GL}_2(\mathbb{Q}_p)$, and following Kubota and Kazhdan–Patterson [18, 19], consider the metaplectic double cover

$$\widehat{\mathrm{GL}}_2(\mathbb{Q}_p) = \mathrm{GL}_2(\mathbb{Q}_p) \times \mu_2$$

defined by the 2-cocycle

$$\sigma(A, B) = \left(\frac{\tau(AB)}{\tau(A)}, \frac{\tau(AB)}{\tau(B)} \det A\right)_p.$$

Its restriction to $G = \mathrm{SL}_2(\mathbb{Q}_p)$ is the cocycle σ_0 of Section 2. The inverse image of G in this cover is therefore $\widetilde{G}_{\sigma_0}$, which we identify with our c-model by the isomorphism ι defined there.

Let $\widetilde{J}_m = (J_m, 1) \in \widehat{\mathrm{GL}}_2(\mathbb{Q}_p)$ and put

$$\alpha_m(g) = J_m g J_m^{-1}.$$

Since $\mathrm{SL}_2(\mathbb{Q}_p)$ is normal in $\mathrm{GL}_2(\mathbb{Q}_p)$, conjugation by $\widetilde{J}_m$ preserves the inverse image of $\mathrm{SL}_2(\mathbb{Q}_p)$. We therefore obtain an automorphism of our cover by

$$\Theta_m = \iota^{-1} \circ \mathrm{Ad}_{\widetilde{J}_m} \circ \iota : \widetilde{G}_c \longrightarrow \widetilde{G}_c.$$

We record explicitly the action of Θ_m, for use in the proof of the following lemma. In the σ-model,

$$\mathrm{Ad}_{\widetilde{J}_m}(g, \epsilon) = (\alpha_m(g), \epsilon r_m(g)),$$

where

$$r_m(g) = \sigma(J_m, g)\sigma(J_m g, J_m^{-1})\sigma(J_m, J_m^{-1}).$$

Since $\tau(J_m) = -p^m$ and $\tau(J_m^{-1}) = 1$, one has

$$\sigma(J_m, J_m^{-1}) = (-p^{-m}, p^m)_p = 1.$$

Applying ι^{-1} yields

$$\Theta_m((g,\epsilon)) = (\alpha_m(g), \epsilon\kappa_m(g)),$$

with

$$\kappa_m(g) = s_p(g)s_p(\alpha_m(g))r_m(g) = s_p(g)s_p(\alpha_m(g))\sigma(J_m,g)\sigma(J_mg,J_m^{-1}).$$

Lemma 5.1. *For $t \in \mathbb{Q}_p$, $u \in \mathbb{Q}_p^\times$, and $\epsilon \in \{\pm 1\}$,*

$$\Theta_m((x(t),\epsilon)) = (y(-p^m t), \epsilon),$$
$$\Theta_m((y(t),\epsilon)) = (x(-p^{-m} t), \epsilon),$$

and

$$\Theta_m((h(u),\epsilon)) = \big(h(u^{-1}), \epsilon(u,p^m)_p\big).$$

In particular, for $u \in \mathbb{Z}_p^\times$,

$$(u,p^m)_p = (u,p)_p^m = \left(\frac{u}{p}\right)^m.$$

Proof. For the upper unipotent subgroup, direct substitution in the GL_2 cocycle gives

$$\sigma(J_m, x(t)) = 1.$$

For $t \neq 0$,

$$\sigma(J_m x(t), J_m^{-1}) = (t, -p^{2m}t)_p = (t,-t)_p = 1,$$

and for $t = 0$ the same value is $\sigma(J_m, J_m^{-1}) = 1$. Moreover

$$s_p(x(t)) = s_p(y(-p^m t)) = 1.$$

Hence $\kappa_m(x(t)) = 1$. The calculation for the lower unipotent subgroup is analogous and gives

$$\sigma(J_m, y(t)) = \sigma(J_m y(t), J_m^{-1}) = 1,$$

with trivial s_p-factors.

For the torus,

$$\sigma(J_m, h(u)) = (u, -1)_p,$$

whereas

$$\sigma(J_m h(u), J_m^{-1}) = (-p^{-m}, p^m u)_p.$$

Using $(-p^{-m}, p^m)_p = 1$, and bilinearity and symmetry,

$$\begin{aligned}(u,-1)_p(-p^{-m},p^m u)_p &= (u,-1)_p(-p^{-m},u)_p \\ &= (u,-1)_p(-1,u)_p(p^m,u)_p = (u,p^m)_p.\end{aligned}$$

Since $s_p(h(u)) = s_p(h(u^{-1})) = 1$, the torus formula follows. For a unit u and odd p one has $(u,p)_p = (\frac{u}{p})$, proving the last assertion. □

Recall

$$\eta((h(u),\epsilon)) = \epsilon\eta_0(u^{-1}), \qquad u \in \mathbb{Z}_p^\times,$$

where η_0 is the underlying quadratic character associated with η, as fixed at the beginning of this section. Define the conjugated character

$$\eta^{J_m}(k) = \eta(\Theta_m^{-1}(k)), \qquad k \in \overline{K_m}.$$

By Lemma 5.1, a direct calculation gives

$$\begin{aligned}\eta^{J_m}((h(u),\epsilon)) &= \eta\big(\Theta_m^{-1}((h(u),\epsilon))\big)\\ &= \eta((h(u^{-1}),\epsilon(u,p^m)_p))\\ &= \epsilon(u,p^m)_p\eta_0(u).\end{aligned}$$

We therefore define $\eta_0^{J_m}$ by

$$\eta_0^{J_m}(u^{-1}) = (u,p^m)_p\eta_0(u),$$

so that

$$\eta^{J_m}((h(u),\epsilon)) = \epsilon\eta_0^{J_m}(u^{-1}).$$

Equivalently, replacing u by u^{-1} and using the fact that η_0 is quadratic,

$$\eta_0^{J_m}(u) = \eta_0(u)(u,p^m)_p = \eta_0(u)\left(\frac{u}{p}\right)^m.$$

Thus

$$\eta_0^{J_m} = \begin{cases}\eta_0, & m \text{ even},\\ \eta_0(\frac{\cdot}{p}), & m \text{ odd};\end{cases}$$

the parity of m enters the formula for the conjugated character.

Let

$$\mathcal{H}_m^{\mathrm{full}}(\eta) = H(\widetilde{G}//\overline{K_m},\eta)$$

denote the full genuine Hecke algebra. Its subalgebra supported in $\overline{K}$ is $H_m(\eta)$, in the notation of Section 2. Since $J_mK_mJ_m^{-1} = K_m$, the automorphism Θ_m preserves $\overline{K_m}$ and induces

$$\Theta_{m,*} : \mathcal{H}_m^{\mathrm{full}}(\eta) \xrightarrow{\sim} \mathcal{H}_m^{\mathrm{full}}(\eta^{J_m}), \qquad (\Theta_{m,*}f)(g) = f(\Theta_m^{-1}(g)).$$

For $\delta \in \{1,\lambda\}$, Lemma 5.1 gives

$$\Theta_m((x(\delta p^{-1}),1)) = (y(-\delta p^{m-1}),1).$$

We next verify the support condition needed to define the corresponding Hecke function.

Lemma 5.2 (Support condition)**.** *For $m \geq 2$ and $\delta \in \mathbb{Z}_p^\times$, the double coset*

$$\overline{K_m}(x(\delta p^{-1}),1)\overline{K_m},$$

supports a unique element of $\mathcal{H}_m^{\mathrm{full}}(\eta)$ whose value at $(x(\delta p^{-1}),1)$ is 1.

Proof. Put $g_\delta = x(\delta p^{-1})$. If

$$k = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in K_m \cap g_\delta K_m g_\delta^{-1}$$

and $k' = g_\delta^{-1}kg_\delta$, then

$$k' = \begin{pmatrix} a-\delta p^{-1}c & b+\delta p^{-1}(a-d)-\delta^2p^{-2}c\\ c & d+\delta p^{-1}c\end{pmatrix}.$$

In particular,

$$d(k')d(k)^{-1} = 1+\delta p^{-1}cd^{-1} \in 1+p\mathbb{Z}_p.$$

Since $c(\eta_0) \leq 1$, this gives $\eta_0(d(k')) = \eta_0(d(k))$.

It remains to check that no central sign is introduced by the chosen lifts. In the σ_0-model, direct substitution gives

$$\sigma_0(k, g_\delta) = \sigma_0(g_\delta, k') = 1.$$

For $c \neq 0$, the compact splitting cochain satisfies

$$\frac{s_p(k')}{s_p(k)} = \left(c, \frac{d(k')}{d(k)}\right)_p = \left(c, 1 + \delta p^{-1} c d^{-1}\right)_p = 1,$$

because $1 + p\mathbb{Z}_p \subset \mathbb{Q}_p^{\times 2}$ for odd p; if $c = 0$, both splitting factors are 1. Using $c(g,h) = \sigma_0(g,h)s_p(g)s_p(h)s_p(gh)$ and $kg_\delta = g_\delta k'$, we conclude that

$$(k, 1)(g_\delta, 1) = (g_\delta, 1)(k', 1)$$

in the cocycle realization used in this paper. Thus the two genuine type characters agree on the intersection. This is precisely the support condition for the character η, and it proves existence and uniqueness of the normalized function. □

We denote this normalized function by

$$\mathcal{X}^{\eta}_{-1,\delta} \in \mathcal{H}^{\mathrm{full}}_m(\eta)$$

and put

$$\mathcal{R}^{\eta}_m = \mathcal{X}^{\eta}_{-1,1} - \mathcal{X}^{\eta}_{-1,\lambda}.$$

The superscript records the character η and will be retained in this section.

Then

$$\Theta_{m,*}(\mathcal{X}^{\eta}_{-1,\delta}) = \mathcal{V}^{\eta^{J_m}}_{m-1,-\delta},$$

where the superscript indicates the conjugated character η^{J_m}. In particular, the conjugated Hecke element belongs to $H_m(\eta^{J_m})$.

5.3. **The Hecke element underlying R_p.** We first record the right coset decomposition that will be used in the comparison.

Lemma 5.3 (Right coset decomposition). *Let $m \geq 2$ and $\delta \in \mathbb{Z}_p^\times$. Then*

$$\begin{aligned} K_m x(\delta p^{-1}) K_m &= \bigsqcup_{u \in \mathbb{Z}_p^\times / \sqrt{1+p\mathbb{Z}_p}} h(u) x(\delta p^{-1}) K_m \\ &= \bigsqcup_{u \in \mathbb{Z}_p^\times / \sqrt{1+p\mathbb{Z}_p}} x(\delta u^2 p^{-1}) K_m. \end{aligned}$$

Here, as in Lemma 2.5, $\sqrt{1 + p\mathbb{Z}_p} = \{u \in \mathbb{Z}_p^\times : u^2 \in 1 + p\mathbb{Z}_p\} = \{\pm 1\}(1 + p\mathbb{Z}_p)$; hence the quotient has $(p-1)/2$ elements.

Proof. Put $g_\delta = x(\delta p^{-1})$ and write

$$k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_m.$$

Then

$$g_\delta^{-1} k g_\delta = \begin{pmatrix} a - \delta p^{-1} c & b + \delta p^{-1}(a - d) - \delta^2 p^{-2} c \\ c & d + \delta p^{-1} c \end{pmatrix}.$$

Because $m \geq 2$ and $c \in p^m\mathbb{Z}_p$, this belongs to K_m if and only if $a \equiv d \pmod p$. Since $ad - bc = 1$ and $c \equiv 0 \pmod p$, this is equivalent to $a \equiv d \equiv \pm 1 \pmod p$. Hence

$$[K_m : K_m \cap g_\delta K_m g_\delta^{-1}] = \frac{p-1}{2}.$$

The reduction map

$$K_m \longrightarrow \mathbb{F}_p^*, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto a \pmod p,$$

is surjective, and the intersection above is precisely the inverse image of $\{\pm 1\}$. Thus the diagonal elements $h(u)$, with $u \in \mathbb{Z}_p^\times/\sqrt{1+p\mathbb{Z}_p}$, give representatives for the right cosets. Finally,

$$h(u)x(t) = x(u^2 t)h(u),$$

which gives the second decomposition. On the corresponding canonical lifts, the cocycle on these products of unit diagonal and upper unipotent elements is trivial, so the same decomposition holds in the metaplectic cover. $\square$

Let (π, V) be a smooth genuine representation and let $v \in V_\eta^{K_m}(\pi)$. On the canonical lifts we have

$$(x(\delta p^{-1}u^2), 1) = (h(u), 1)(x(\delta p^{-1}), 1)(h(u)^{-1}, 1),$$

so biequivariance and the normalization $\mathcal{X}^\eta_{-1,\delta}((x(\delta p^{-1}), 1)) = 1$ give $\mathcal{X}^\eta_{-1,\delta}((x(\delta p^{-1}u^2), 1)) = 1$. Thus, with $\mathrm{vol}(\overline{K_m}) = 1$, the Hecke action defined in Section 2 and Lemma 5.3 give

$$\mathcal{X}^\eta_{-1,\delta} v = \sum_{u \in \mathbb{Z}_p^\times/\sqrt{1+p\mathbb{Z}_p}} \pi((x(\delta p^{-1}u^2), 1))v.$$

Since $(x(b), 1)$ fixes v for $b \in \mathbb{Z}_p$, each summand depends only on δu^2 modulo p. The values u^2 run through the nonzero squares modulo p, while λu^2 runs through the nonsquares. Subtracting the two sums therefore yields

$$\mathcal{R}^\eta_m v = \sum_{a \in \mathbb{F}_p^*} \left(\frac{a}{p}\right) \pi((x(a/p), 1))v.$$

With the additive normalization $\mathrm{vol}(\mathbb{Z}_p) = 1$ used in (6), comparison with the preceding formula gives

$$\mathcal{R}^\eta_m v = pR^\pi_p v, \qquad v \in V_\eta^{K_m}(\pi).$$

5.4. **From R_p to $\mathcal{W}_{m-1}$.** Applying $\Theta_{m,*}$ gives

$$\Theta_{m,*}(\mathcal{R}^\eta_m) = \mathcal{V}^{\eta^{J_m}}_{m-1,-1} - \mathcal{V}^{\eta^{J_m}}_{m-1,-\lambda}.$$

By the square-class dependence noted in Section 2, if $\left(\frac{-1}{p}\right) = 1$, then

$$\mathcal{V}^{\eta^{J_m}}_{m-1,-1} = \mathcal{V}^{\eta^{J_m}}_{m-1,1}, \qquad \mathcal{V}^{\eta^{J_m}}_{m-1,-\lambda} = \mathcal{V}^{\eta^{J_m}}_{m-1,\lambda},$$

whereas if $\left(\frac{-1}{p}\right) = -1$, the two operators are interchanged. Consequently

$$\Theta_{m,*}(\mathcal{R}^\eta_m) = \left(\frac{-1}{p}\right) \mathcal{W}^{\eta^{J_m}}_{m-1}.$$

Combining this identity with the preceding comparison gives the following form of the relation with Ishimoto's operator.

Proposition 5.4 (Hecke algebra comparison with Ishimoto)**.** *Let $m \geq 2$, and let η be the genuine extension to $\overline{K_m}$ of either the trivial or the nontrivial quadratic character modulo p. Put*

$$\mathcal{R}_m^\eta = \mathcal{X}_{-1,1}^\eta - \mathcal{X}_{-1,\lambda}^\eta \in \mathcal{H}_m^{\mathrm{full}}(\eta).$$

Then

$$\Theta_{m,*}(\mathcal{R}_m^\eta) = \left(\frac{-1}{p}\right) \mathcal{W}_{m-1}^{\eta^{J_m}}.$$

Let (π, V) be a genuine smooth representation of $\widetilde{G}_c$ and define $\pi^{J_m} = \pi \circ \Theta_m^{-1}$. The identity map on the underlying vector space V identifies

$$V_\eta^{K_m}(\pi) = V_{\eta^{J_m}}^{K_m}(\pi^{J_m}).$$

Under this identification,

$$pR_p^\pi v = \left(\frac{-1}{p}\right) \pi^{J_m}\left(\mathcal{W}_{m-1}^{\eta^{J_m}}\right) v, \qquad v \in V_\eta^{K_m}(\pi).$$

Proof. The formula for $\Theta_{m,*}(\mathcal{R}_m^\eta)$ follows from the preceding calculation, and the equality of the fixed-type spaces follows from the definitions. Since Θ_m preserves $\overline{K_m}$, it preserves the Haar measure normalized by $\mathrm{vol}(\overline{K_m}) = 1$. A change of variables in the Hecke action therefore gives

$$\pi^{J_m}(\Theta_{m,*}F) = \pi(F), \qquad F \in \mathcal{H}_m^{\mathrm{full}}(\eta).$$

Apply this to $F = \mathcal{R}_m^\eta$ and use $\mathcal{R}_m^\eta v = pR_p^\pi v$ to obtain the asserted formula. □

For odd m, this compares the actions in the conjugated representation and conjugated type; no identification with the original (π, η)-space is asserted.

Assume that m is even and set

$$g_m := p^{-m/2} J_m = \begin{pmatrix} 0 & p^{-m/2} \\ -p^{m/2} & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p),$$

since $\det(J_m) = p^m$. Put $z = p^{m/2}$, so that $J_m = z g_m$. In the σ-model, direct computation gives

$$\widetilde{z}(g, \epsilon) = (g, \epsilon)\widetilde{z}, \qquad \widetilde{z} = (zI, 1), \quad g \in \mathrm{SL}_2(\mathbb{Q}_p), \quad \epsilon \in \mu_2.$$

Thus $\widetilde{z}$ centralizes $\widetilde{G}_{\sigma_0}$. Since $\widetilde{J}_m$ and $\widetilde{z}(g_m, 1)$ differ only by an element of the central kernel, they induce the same conjugation, and hence

$$\mathrm{Ad}_{\widetilde{J}_m} = \mathrm{Ad}_{(g_m,1)} \quad \text{on } \widetilde{G}_{\sigma_0}.$$

Now put $\widetilde{g}_m = (g_m, 1) \in \widetilde{G}_c$. The lower-right entry of g_m is zero, so $s_p(g_m) = 1$ and $\iota(\widetilde{g}_m) = (g_m, 1)$. Consequently

$$\Theta_m = \mathrm{Ad}_{\widetilde{g}_m}.$$

In particular, $\pi^{J_m} \simeq \pi$: with $A_m = \pi(\widetilde{g}_m)$, we have

$$A_m \pi^{J_m}(\widetilde{g}) A_m^{-1} = \pi(\widetilde{g}), \qquad \widetilde{g} \in \widetilde{G}_c.$$

As $\eta^{J_m} = \eta$, the operator A_m preserves $V_\eta^{K_m}(\pi)$. Proposition 5.4 therefore gives

$$pA_m R_p^\pi A_m^{-1} = \left(\frac{-1}{p}\right) \pi(\mathcal{W}_{m-1}^\eta) \quad \text{on } V_\eta^{K_m}(\pi).$$

The factor $p^{-m/2}$ merely rescales the conjugating matrix; it is not a normalization of either R_p or $\mathcal{W}_{m-1}$.

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